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PREFACE.

THE following pages are intended to form a text-book on the subject suitable for beginners, and they extend as far as is required at Cambridge in the preliminary examination of candidates for mathematical honours.

It is difficult to define the exact meaning of "geometrical," but I think that any process would be universally recognised as strictly geometrical, if every expression and idea involved had a direct geometrical interpretation, and had no necessary reference to any units of measurement. I have only departed from this test in one or two cases, where custom has sanctioned a more lax interpretation, or where an argument has been very considerably simplified. At the same time, I have fully recognised the fact that the idea of ratio, and propositions on ratio and proportion, are in no sense geometrical, though of course true for geometrical as for other quantities; and as practically boys become acquainted with the properties of proportionals by means of arithmetic and algebra, I have introduced these properties in a form in which they will be most readily appreciated and understood. Consequently, although I refer to Euclid when using the geometrical theorems on the straight line and circle, I never quote the fifth book of the Elements; and in order to make the argument clearer to such students as I am contemplating, I invariably use the fractional notation to indicate a ratio instead of the customary sign (:). It will, I expect, be found that in long propositions the argument may thus be followed much more easily, while at the same time the processes are not a whit less geometrical.

In order that the ellipse and hyperbola may be studied simultaneously, chapters III. and IV. are printed together, the former on the even, the latter on the odd pages, and in such a manner that the corresponding propositions for the two curves face each other. The properties of the asymptotes of the hyperbola naturally cause a divergence, but it has not been found necessary to depart from this arrangement in the later portions of these chapters, inasmuch as the propositions peculiar to the asymptotes occupy about the same space as those which treat of projections, which are introduced into the ellipse at the same time.

Chapters II., V., and VI. had perhaps better be omitted till the student has mastered the remainder of the book, and may be acquired gradually whilst studying analytical geometry and working problems.

Besides the problems at the ends of the chapters, a considerable number, on account either of their importance or extreme simplicity, are incorporated with the text.

I have to thank the Rev. C. Taylor, who has done so much for this subject, for kindly allowing me to use the proof of Prop. 7, chap. III., IV., the Rev. W. Allen Whitworth, for the proof of prop. 6, chap. III., IV., and several other friends for suggestions or assistance in revising the proof sheets; and I expect that both the text and problems will be found to be extremely free from errors.

G. R.

WINCHESTER, *June* 1873.

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CONIC SECTIONS.

INTRODUCTION.

Definition 1. A *Conic Section*, or more briefly, a *Conic*, is a plane curve traced out by a point, which moves in such a manner, that its distance from a fixed point bears a constant ratio to its perpendicular distance from a fixed straight line.

The fixed point is called the *Focus*, and the fixed straight line the *Directrix* of the Conic. The constant ratio is called the *Eccentricity.*

If the eccentricity is unity, that is, if the distance of every point on the curve from the focus is equal to its distance from the directrix, the conic is called a *Parabola.*

If the eccentricity is less than unity, that is, if the distance of every point on the curve from the focus is less than its distance from the directrix, the conic is called an *Ellipse.*

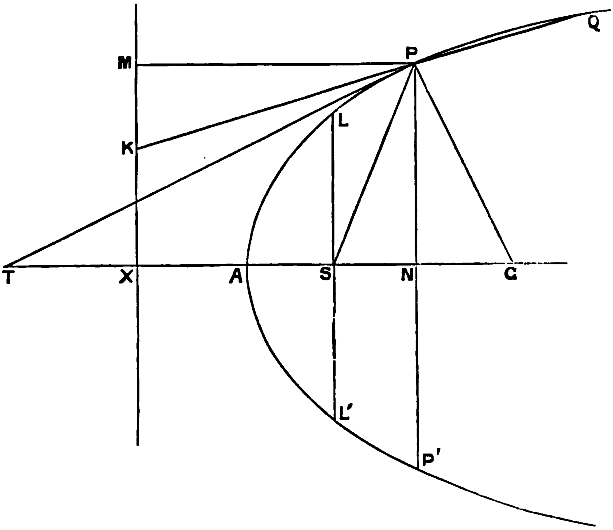
If the eccentricity is greater than unity, that is, if the distance of every point on the curve from the focus is greater than its distance from the directrix, the conic is called a *Hyperbola.*

Let S be a fixed point, and MX a fixed straight line, then the curve described in the figure will be a conic if the ratio of the distance SP of a point P on it from S, to the distance PM, drawn perpendicular to MX, be the same wherever P may be taken; and the curve will be a Parabola, Ellipse, or Hyperbola, according as this ratio is equal to, less than, or greater than unity (as $SP = <$ or $> PM$).

The point S and the straight line MX are respectively the focus and directrix of the conic.

Def. 2. The line SX drawn through the focus perpendicular to the directrix is called the *Axis* of the conic.

Def. 3. Let A be a point in SX such that the ratio of AS to AX is equal to the eccentricity, then A is a point on the conic, and is called the *Vertex*.



In order that the student may obtain some general notions of the nature of the curves, the properties of which we are about to investigate, it will be found a useful exercise to try to draw some different conics by assuming special values for the eccentricity. Thus, if a point and a straight line be chosen for focus and directrix, it is easy, by means of the eye or a pair of compasses, to obtain a number of points which are equidistant from these, and thus roughly to determine the form of a parabola. Or again, by assuming the eccentricity to be $\frac{1}{2}$ or 2, an ellipse or hyperbola may be sketched out. In particular, it ought to be noticed where the curve cuts the axis. It will be easily seen that the parabola cuts the axis at one point only, while the ellipse or hyperbola meets it at two points; in the former case on opposite sides of the focus, but on the same side of the directrix; in the latter, on the same side of the focus, but on opposite sides of the directrix.

NOTE. It is evident that the axis divides the conic symmetrically, that is, that the portions of the curve above and below this line are exactly equal and similar.

DEF. 4. The line PN drawn from any point P on the conic, perpendicular to the axis, is called the *Ordinate* of the point P; and the portion AN of the axis intercepted between the ordinate and the vertex is called the *Abscissa.*

If PN be produced to meet the curve again at P', $P'N$ is equal to PN (Note), and PP' is called the *Double Ordinate.*

DEF. 5. The double ordinate LSL' through the focus is called the *Latus Rectum.*

DEF. 6. Any straight line, as KPQ, drawn so as to cut the conic, is called a *Secant*, and the portion PQ of the secant intercepted by the curve is called a *Chord* of the conic.

DEF. 7. Any chord drawn through the focus is called a *Focal Chord.*

The distance of any point from the focus is called the *Focal Distance* of that point.

DEF. 8. If the point Q be supposed to move gradually along the curve up to P, the secant QPK, in its limiting position, PT, when Q ultimately coincides with P, is said to be the *Tangent* at P.

DEF. 9. The line PG drawn at right angles to the tangent at P is called the *Normal* at the point P.

DEF. 10. The portion, TN, of the axis intercepted between the tangent and ordinate of any point is called the *Subtangent;* and the portion of the axis, GN, intercepted between the normal and the ordinate is called the *Subnormal.*

The letters used in the figure ought to be observed, as we shall in the following pages use, as far as possible, the same letters for the same points; and it will not be deemed necessary to describe the figure fully in every proposition. Thus it must be understood that if the point S is mentioned, the focus is meant; if A, the vertex; if PG, the normal at P, etc.

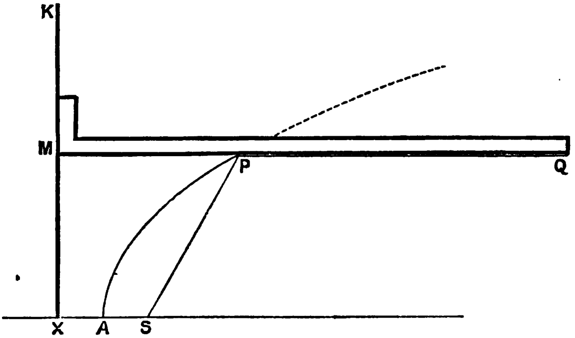
CHAPTER I.

THE PARABOLA.

DEF. 11. A Parabola is a plane curve, traced out by a point, which moves in such a manner, that its distance from a fixed point is always equal to its perpendicular distance from a fixed straight line.

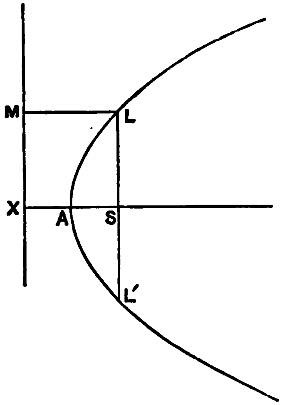
HOW TO DRAW A PARABOLA.

Let *MQ* be a straight rod or bar whose extremity, *M*, is capable of sliding along a fixed straight line *KX*, whilst the bar always remains at right angles to this line; and let one end of a string, whose length is equal to that of the bar, be fastened to the further extremity, *Q*, of the bar, and the other end of the string be fastened to a fixed point *S;* then a pencil, *P*, which keeps the string tight and always remains in contact with the bar, will, as *M* moves along *XK*, trace out a portion of a parabola. For in every position of the bar $SP=PM$.



PROPOSITION I.

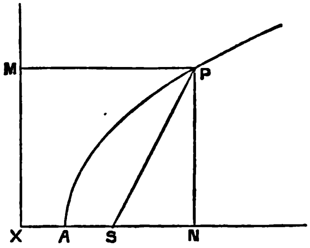
The Latus Rectum $LL'=4AS$.



Draw LM perpendicular to the directrix.
Then $SL=LM$ (Def. 11)
$=SX$ (Euc. I. 34).
$=2AS$,
(since $AS=AX$, Def. 11).
But $LL'=2SL$ (Note, p. 3).
$\therefore LL'=4AS$.

PROPOSITION II.

The ordinate of any point is a mean proportional between the latus rectum and the abscissa of that point. ($PN^2=4AS.AN.$)

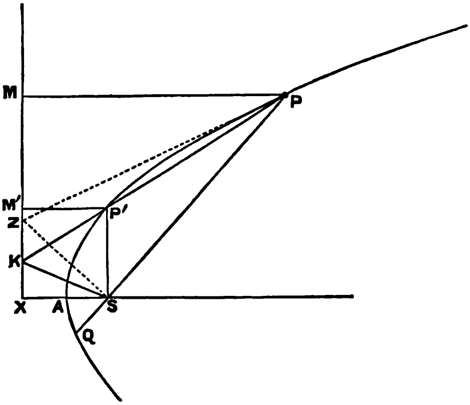


Join SP, and draw PM perpendicular to the directrix.
Then $PN^2=SP^2-SN^2$ (Euc. I. 47).
$=XN^2-SN^2$ (Def. 11, and Euc. I. 34).
$=(AN+AS)^2-(AN-AS)^2$, since $AX=AS$.
$=4AN.AS$ (Smith's Euc. II. B).
It may be similarly proved when N is between A and S.

Proposition III.

The tangent at any point bisects the angle between the focal distance and the perpendicular on the directrix; and the portion of the tangent intercepted between the directrix and the point of contact subtends a right angle at the focus.

(PZ bisects the angle SPM; and PSZ is a right angle.)



Let P' be a point on the curve near P. Draw the chord PP', and produce it to meet the directrix at K. Draw PM, $P'M'$ perpendicular to the directrix; join SP, SP', SK, and produce PS to Q.

Then since the triangles KMP, $KM'P'$ are similar,

$$\frac{KP}{KP'}=\frac{MP}{M'P'} \text{ (Euc. VI. 4)}$$

$$=\frac{SP}{SP'} \text{ (Def. 11)}$$

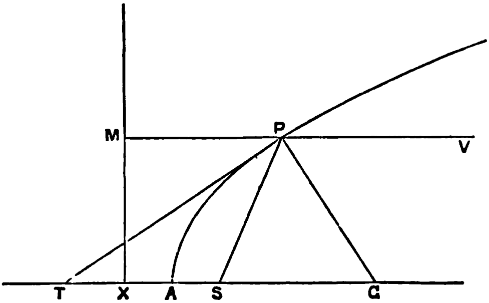
$\therefore$ SK bisects the angle $P'SQ$ (Euc. VI. A).

Now if P' moves along the curve up to P, $PP'K$ becomes ultimately the tangent at P (Def. 8), and the angle PSP' becomes indefinitely small, while the angle $P'SQ$ approaches two right angles; and therefore the angle $P'SK$, which is always half of $P'SQ$, becomes ultimately a right angle. Hence if PZ is the tangent, PSZ is a right angle.

Again, since $SP=MP$, and PZ is common to the two triangles SPZ, MPZ, and the angles at S and M are right angles, the two triangles are equal in all respects (Smith's Euc. I. E, Cor.), and angle $SPZ=MPZ$, that is, PZ bisects the angle SPM.

Cor. 1. The fact that SK bisects the angle $P'SQ$ does not depend on P' being near P, and is true for any secant of the parabola.

Cor. 2. Since the tangent at the vertex A must by the proposition bisect the angle between AS and AX, it follows that it is at right angles to the axis.



Cor. 3. If PG be the normal at P, and MP be produced to V, then PG bisects the angle SPV. For TPG is a right angle, $\therefore$ VPG is the complement of MPT, and SPG the complement of SPT; but $MPT=SPT$ $\therefore$ $VPG=SPG$.

Cor. 4. Since MPV is parallel to the axis, it follows that angle $STP=MPT=SPT$; and also that $SGP=GPV=SPG$. Hence $ST=SP=SG$ (Euc. I. 6).

Examples.

1. Prove that, for the point on the parabola, for which the abscissa is equal to the ordinate, each is equal to the latus rectum.

2. Show that the tangents at the extremities of the latus rectum meet the axis at the foot of the directrix.

3. Show how to draw a tangent to the parabola from a given point in the axis.

4. If in the last figure MT and MS be joined, prove that $MPST$ is a rhombus, and that $MPGS$ is a parallelogram.

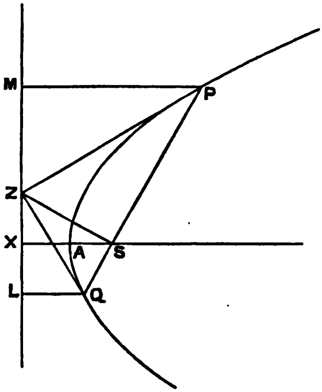
5. What is the least possible value of TG?

6. Having given two points on a parabola and its directrix, find the focus.

7. Having given two points on a parabola and its focus, find the directrix.

Proposition IV.

Tangents at the extremities of a focal chord intersect at right angles on the directrix.



Let PSQ be a focal chord.

Draw SZ at right angles to PQ.

Then the tangent at P meets the directrix where SZ meets it, and also the tangent at Q (Prop. 3).

Again $SP=PM$, and PZ is common to the two triangles SPZ and MPZ, and the angle $SPZ=MPZ$ (Prop. 3).

$\therefore$ angle $SZP=MZP$.

Similarly $SZQ=LZQ$.

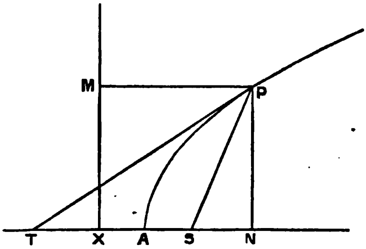
$\therefore$ angle $PZQ=MZP+LZQ$.

But $PZQ+MZP+LZQ=$two right angles (Euc. I. 13).

$\therefore$ PZQ is a right angle.

Proposition V.

The subtangent is equal to twice the abscissa. ($NT=2AN$.)



Draw PM perpendicular to the directrix.

Then $ST=SP$ (Prop. 3, Cor. 4)

$=PM=XN$.

That is, $SA+AT=XA+AN$.

But $SA=XA$

$\therefore AT=AN$

$\therefore NT=2AN$.

Proposition VI.

The subnormal is constant and equal to half the latus rectum. ($NG=2AS$.)

Draw PM perpendicular to the directrix,

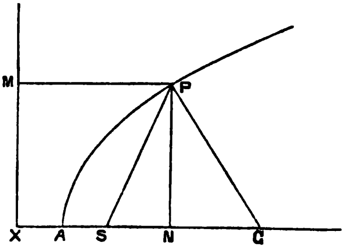
Then $SG = SP$ (Pr. 3. Cor. 4)

$= PM=XN$,

that is $SN+NG=XS+SN$

$\therefore NG=XS$

$=2AS$.

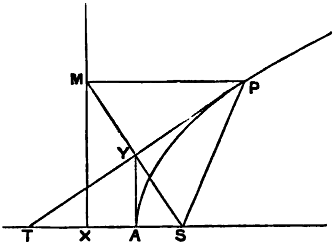


Example.

Determine the point on a parabola whose subtangent is equal to its subnormal.

PROPOSITION VII.

If a perpendicular be drawn from the focus on a tangent, the point of intersection lies on the tangent at the vertex.



Let PT be the tangent at P. Draw PM perpendicular to the directrix, and join SM, cutting PT at Y.

Then $SP=PM$, and PY is common to the two triangles SPY, MPY, and angle $SPY=MPY$; $\therefore$ the two triangles are equal in all respects, and $\therefore$ SY is at right angles to PYT, and $SY=MY$. Join AY.

Then since $SY=MY$, and $SA=AX$

$\therefore$ AY is parallel to XM (Euc. VI. 2),

that is, AY is perpendicular to the axis, and is therefore the tangent at the vertex (Prop. 3, Cor. 2).

COR. 1. Since $SP=ST$ (Prop. 3, Cor. 4), it is evident that PT is bisected at Y.

COR. 2. Since SYT is a right-angled triangle, and YA is perpendicular to ST, it follows that the triangles SAY, YAT, SYT, SYP, are all similar (Euc. VI. 8), and $\therefore$ $SY^2=AS.ST=AS.SP$; that is, SY is a mean proportional between AS and SP.

EXAMPLE.

If PN be the ordinate of P, prove that $NY=PY$.

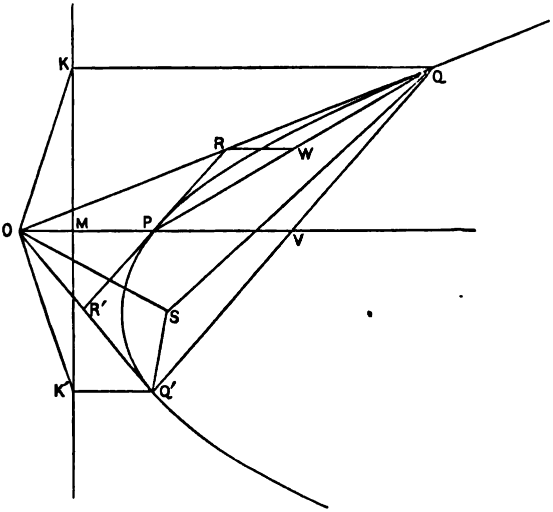
DEF. 12. Any line parallel to the axis of a parabola is called a *Diameter*.

The portion of any line, parallel to the tangent at the extremity of a diameter, intercepted between the parabola and the diameter, is called an *Ordinate* of that diameter.

The portion of the diameter intercepted between the parabola and an ordinate is called an *Abscissa*.

PROPOSITION VIII.

If from a point O, a pair of tangents OQ, OQ′ be drawn to a parabola, and the diameter OPV be drawn meeting the curve at P and the chord QQ′ at V; then shall QQ′ and OV be bisected respectively at V and P, and the tangent at P shall be parallel to QQ′.



Let RPR' be the tangent at P, and let OP meet the directrix at M. Draw QK, $Q'K'$ perpendicular to the directrix, and join OK, OK', SO, SQ, SQ', and PQ. Draw the diameter RW meeting PQ at W.

Then because $SQ = QK$, and OQ is common to the two triangles SQO and KQO, and the angle $SQO = KQO$ (Prop. 3).

$\therefore$ $OK = OS$ (Euc. I. 4). Similarly it may be shown that $OK' = OS$ $\therefore$ $OK = OK'$.

$\therefore$ $KM = K'M$ (since OM is perpendicular to KK').

$\therefore$ $QV = Q'V$ (since QK, VM, and $Q'K'$ are parallel). (See Euc. VI. 10; or it may be regarded as a particular case of Euc. XI. 17.)

Again, since RW is a diameter it might be proved in the same manner that $QW=PW$.

$\therefore$ $QR=OR$ (since RW is parallel to OP).

Similarly $Q'R'=OR'$.

$\therefore$ RPR' is parallel to QQ';

and $\therefore$ $OP=PV$ (Euc. VI. 2).

Cor. 1. From this it is manifest that if any number of parallel chords be drawn in a parabola, their middle points will all lie on the diameter which passes through the point of contact of the tangent drawn parallel to the chords.

A diameter is by some writers defined as the locus of the middle points of a series of parallel chords.

Cor. 2. It is evident that RR' is bisected at P, and that $QV=2PR$.

Cor. 3. Since $OS=OK=OK'$, we have an easy method of drawing tangents from any point O to the parabola.

With centre O and distance OS describe a circle cutting the directrix at K, K': draw KQ, $K'Q'$ parallel to the axis, cutting the curve at Q, Q'; then OQ, OQ' are evidently tangents.

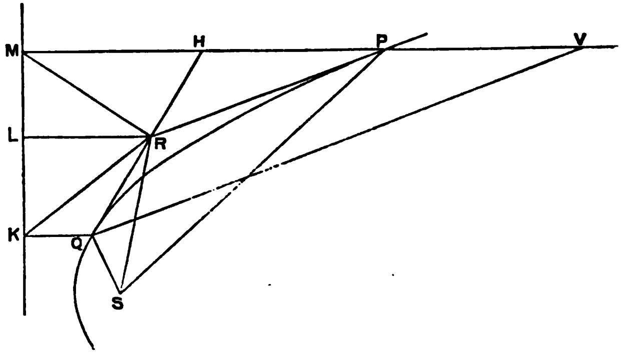
Proposition IX.

The ordinate of the diameter through any point on a parabola is a mean proportional between the corresponding abscissa and four times the focal distance of that point. ($QV^2=4SP.PV.$)

Let MPV be the diameter through P, and QV any ordinate; let the tangents at P and Q intersect at R, and produce QR to meet PM at H. Draw QK, RL perpendicular to the directrix, and join SP, SQ, SR, MR, and KR.

Then because $SP=PM$, and PR is common to the two triangles SPR, MPR, and angle $SPR=MPR$, $\therefore$ the triangles SPR and MPR are equal in all respects, and $MR=SR$; also

SQR and KQR are equal, and $KR=SR$: hence $MR=KR$, $\therefore LR$ bisects MK, and also the angle MRK.



Again, since the three angles SRP, KRQ, KRL, are equal respectively to MRP, SRQ, MRL, and that these six angles are together equal to four right angles (Euc. I. 15, Cor. 2),

$\therefore SRP+KRQ+KRL=$two right angles.

$\therefore SRP+KRQ+RKQ=$two right angles (Euc. I. 29).

But $KQR+KRQ+RKQ=$two right angles (Euc. I. 32).

$\therefore SRP=KQR$

$=RHP$ (Euc. I. 29).

Also, angle $SPR=RPH$; $\therefore$ the triangles SPR and RPH are similar;

$$\therefore \frac{RP}{SP}=\frac{HP}{RP} \text{ (Euc. VI. 4),}$$

or $RP^2=SP.HP$.

Again, since $ML=LK$, and MH, LR, and KQ are parallel,

$\therefore HR=RQ$.

$\therefore HP=PV$, and $QV=2RP$ (since QV and RP are parallel);

$\therefore QV^2=4RP^2$.

But $4RP^2=4SP.HP$;

$\therefore QV^2=4SP.PV$.

PROPOSITION X.

The two tangents drawn from a given point to a parabola subtend equal angles at the focus; and the focal distance of the given point is a mean proportional between the focal distances of the points of contact. (Angle $PSR = QSR$, and $SR^2 = SP.SQ$.)

Let R be the given point, RP, RQ the two tangents (see Fig. to Prop. 9).

Then it may be shown, as in Prop. 9, that

angle $PSR = PMR = MRL = LRK = RKQ = QSR$.

Also, as in Prop. 9,

angle $SRP = KQR = SQR$;

$\therefore$ angle $SRQ = RPS$ (Euc. I. 32), and the two triangles, SRP, SQR are similar;

$$\therefore \frac{SR}{SP} = \frac{SQ}{SR},$$

$$\text{or } SR^2 = SP.SQ.$$

DEF. 13. The double ordinate of a diameter, drawn through the focus, is called the *Parameter* of that diameter.

PROPOSITION XI.

The parameter of the diameter PV = 4SP.

Let $QVSQ'$ be the parameter of PV. Draw the tangent PT at P, and join SP.

Since PT makes equal angles with SP and PV (Prop. 3), and SQ is parallel to PT,

$\therefore$ angle $PVS = PSV$ (Euc. I. 29).

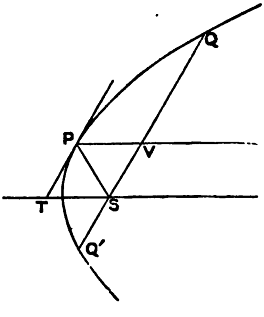
$\therefore$ $PV = SP$ (Euc. I. 6).

But $QV^2 = 4SP.PV$ (Prop. 9),

$\therefore$ $QV^2 = 4SP^2$,

$\therefore$ $QV = 2SP$,

$\therefore$ $QQ' = 4SP$ (Prop. 8).



EXAMPLES ON THE PARABOLA.

1. Show that the focal distance of any point on the parabola is greater than its abscissa by one-fourth of the latus rectum.

2. If P be a point on a parabola such that $SP=3AS$, then AP is bisected by the latus rectum.

3. Find the locus of the centres of the circles which pass through a given point, and touch a given straight line.

4. Find the locus of the centres of the circles which touch a given circle and a given straight line.

5. Prove that the least focal chord which can be drawn in a parabola is the latus rectum.

6. Having given the focus, determine a parabola touching a given straight line at a given point.

7. If PNQ be a double ordinate of a parabola, such that APQ is an equilateral triangle, prove that AN is three times the latus rectum.

8. If PN, the ordinate to a parabola at the point P, be equal to the latus rectum, show that N is the foot of the normal at a point whose abscissa is one-half the latus rectum.

9. A circle is drawn touching a parabola at its vertex, the diameter of the circle being equal to the latus rectum of the parabola; prove that the length of the tangent drawn from any point on the parabola to the circle is equal to the abscissa of that point. Hence show that the circle lies entirely within the parabola.

10. A circle is described on the latus rectum of a parabola as a diameter, and any chord QPN is drawn parallel to the axis of the parabola, meeting the circle at Q, the parabola at P, and the latus rectum at N: prove that QN^2 varies as PN.

11. Through any point P of a parabola, a straight line QPQ' is drawn parallel to the latus rectum and terminated by the tangents at its extremities, and PR is drawn perpendicular to the latus rectum produced if necessary; show that $PR^2=PQ.PQ'$.

38. Prove that the normals at the extremities of any focal chord of a parabola intersect on the diameter which bisects the chord.

39. Prove that the diameter at one extremity of a focal chord of a parabola bisects the portion of the normal at the other extremity intercepted by the curve.

40. If the chord QQ' of a parabola meet the axis at N, and if from O, the point of intersection of the tangents at Q, Q', OK be drawn perpendicular to the directrix, prove that $OKNS$ is a parallelogram.

41. Find the locus of the middle points of all the focal chords of a parabola.

42. If QV be an ordinate of the diameter PV, and QD be drawn perpendicular to that diameter, prove that $QD^2=4AS.PV$.

43. A parabola being given in position, show how to determine its directrix and focus.

44. If QV be any ordinate to the diameter through a point P of a parabola, and $Q'V'$ be the ordinate through the focus, show that the tangent drawn from P to any circle passing through V, V' is equal to one-half of QV.

45. A focal chord to a parabola is bisected at V, and a line at right angles to it through V meets the axis at F; show that SF is half the length of the chord.

46. If from a point O a pair of tangents OQ, OQ' be drawn to a parabola, prove that a perpendicular to OS through S will meet QQ' at the directrix.

47. TP, TQ are two tangents to a parabola at P and Q; show that the triangles SPT, SQT are as SP to SQ.

48. If AQ be a chord of a parabola through the vertex A, and QR be drawn perpendicular to AQ to meet the axis at R, prove that AR will be equal to the chord through the focus parallel to AQ.

49. The angle between the focal distances of two points on a parabola is double the acute angle between the tangents at those points.

50. TP, TQ are two tangents to a parabola: show that TP makes the same angle with ST, that TQ makes with the axis.

CHAPTER II.

MISCELLANEOUS PROPOSITIONS ON THE PARABOLA.

Proposition XII.

Every straight line drawn at right angles to the directrix meets the parabola at one point, and one point only.

Let MPQ be a line at right angles to the directrix. Join SM, and make the angle MSP equal to SMP.

Then $PS=PM$, and $\therefore$ P is on the curve.

Also, if possible, let the point Q, on MP produced, be on the parabola; then

$$\begin{aligned} SQ=QM&=PQ+PM \\ &=PQ+SP, \end{aligned}$$

which is absurd (Euc. I. 20).

Similarly it may be shown that no point on MP on the other side of P can be on the curve.

Proposition XIII.

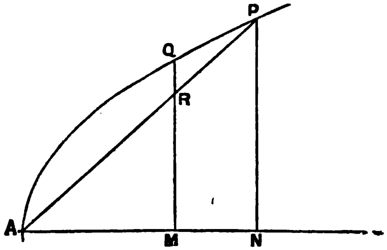
No straight line can meet the parabola at more points than two.

For, if possible, let a straight line meet the parabola at P, Q, R, and let it meet the directrix at K. Join SK, SP, SQ, SR. Then SK bisects the exterior angle between SP and SQ, and also bisects the exterior angle between SP and SR (Prop. 3, Cor. 1), which is impossible.

Proposition XIV.

The parabola is concave to its axis.

Let P be any point on the parabola: join AP. Then it will be sufficient to prove that AP lies between the curve and the axis, or that, if the ordinate QM be drawn from any point between A and P, meeting AP at R, QM is greater than RM.



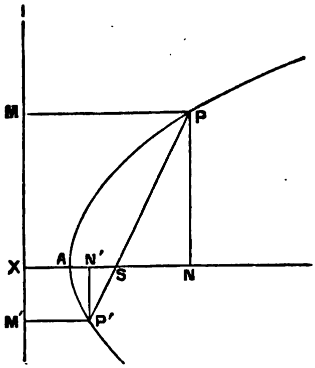
For $\frac{RM^2}{AM^2}=\frac{PN^2}{AN^2}=\frac{4AS.AN}{AN^2}=\frac{4AS}{AN}$.

Also $\frac{QM^2}{AM^2}=\frac{4AS.AM}{AM^2}=\frac{4AS}{AM}$.

From which it follows, since AN is greater than AM, that QM is greater than RM.

Proposition XV.

The semi-latus rectum is a harmonic mean between the segments of any focal chord.



Let PSP' be a focal chord: then $\frac{1}{SP}+\frac{1}{SP'}=\frac{1}{AS}$.

Draw the ordinates PN, $P'N'$; and also $PM, P'M'$ perpendicular to the directrix.

Then $\frac{SN}{SP}=\frac{SN'}{SP'}$ (Euc. VI. 4),

or $\frac{XN-XS}{SP}=\frac{XS-XN'}{SP'}$

$\therefore \frac{SP-XS}{SP}=\frac{XS-SP'}{SP'}$ (Euc. I. 34 and Def. 11);

$\therefore 1-\frac{2AS}{SP}=\frac{2AS}{SP'}-1$;

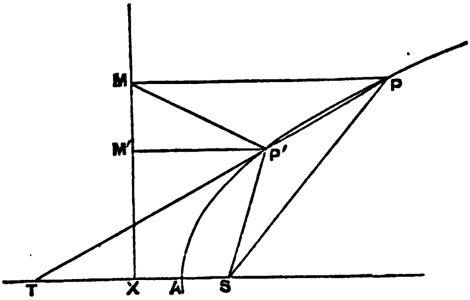
$\therefore 2=\frac{2AS}{SP}+\frac{2AS}{SP'}$; $\therefore \frac{1}{AS}=\frac{1}{SP}+\frac{1}{SP'}$.

PROPOSITION XVI.

The straight line PT, which bisects the angle between the focal distance SP and the perpendicular PM on the directrix, does not meet the parabola at any other point.

For, if possible, let it meet the curve again at P'.

Draw $P'M'$ perpendicular to the directrix, and join SP', $P'M$.



Then in the triangles SPP', MPP',

$SP=PM$, angle $SPP'=MPP'$, and PP' is common;

$\therefore SP'=P'M$. But $SP'=P'M'$ (Def. 11); $\therefore P'M=P'M'$;

$\therefore$ angle $P'M'M=P'MM'$ (Euc. I. 5), which is impossible, since $P'M'M$ is a right angle (Euc. I. 17).

$\therefore PT$ meets the curve at one point only; and since it bisects the angle SPM, it is the tangent at P (Prop. 3).

$\therefore$ the tangent meets the curve at one point only.

Proposition XVII.

If T be any point on the tangent at P, and TK, TF be drawn perpendicular to the directrix, and the focal distance SP respectively, then $SF=TK$.

Draw PM perpendicular to the directrix, and let PT meet the directrix at Z. Join SZ, which is at right angles to SP (Prop. 3). Then from similar triangles,

$$\frac{TK}{PM}=\frac{TZ}{PZ}=\frac{SF}{SP} \text{ (Euc. VI. 4).}$$

But $PM=SP$, $\therefore TK=SF$.

Proposition XVIII.

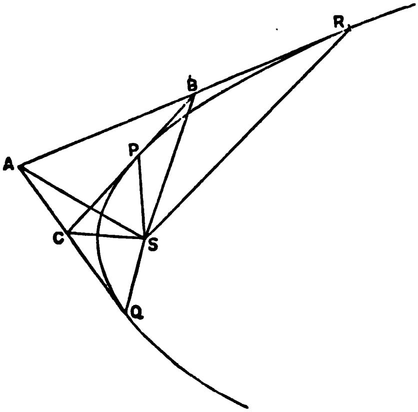
If from G, the foot of the normal at P, GE be drawn perpendicular to SP, then $PE=2AS$.

Draw the ordinate PN; then in the triangles GEP, PNG, the angle $EPG=NGP$ (Prop. 3, Cor. 4), the angles at N and E are right angles, and PG is common;

$$\therefore PE=NG=2AS \text{ (Prop. 6).}$$

Proposition XIX.

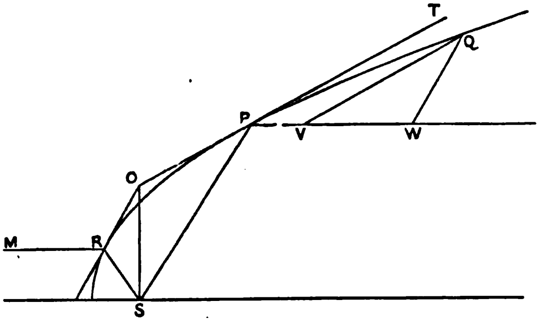
If a parabola touch three straight lines, its focus will lie on the circle circumscribing the triangle formed by the three lines.



Let *BC*, *CA*, *AB* touch the parabola at *P*, *Q*, *R* respectively. Join these points with the focus *S*. Then angle $SBP=SRB$ (since the triangles *SPB*, *SBR* are similar, Prop. 10). Similarly $SAQ=SRB$; $\therefore$ $SBP=SAQ$, and $\therefore$ a circle described round *SBC* will pass through *A* (Euc. III. 21).

Proposition XX.

If QV be an ordinate of the diameter PV, and QW be drawn parallel to the tangent at any point R to meet PV at W, then $QW^2=4SR.PV$.



Let the tangents at *R* and *P* meet at *O*, and produce *OP* to *T*. Draw *RM* parallel to the axis, and join *SP*, *SO*, *SR*.

Then the angle $SOR=SPO$ (Prop. 10)
$=TPV$ (Prop. 3)
$=QVW$ (Euc. I. 29).

Again, angle $SRO=MRO$ (Prop. 3)
$=QWV$ (since *MR* is parallel to *VW* and *OR* to *QW*); $\therefore$ the two triangles *SRO* and *QWV* are similar;

$$\therefore \frac{QW^2}{QV^2}=\frac{SR^2}{SO^2};$$

$$\therefore \frac{QW^2}{4SP.PV}=\frac{SR^2}{SR.SP} \text{ (Props. 9 and 10)};$$

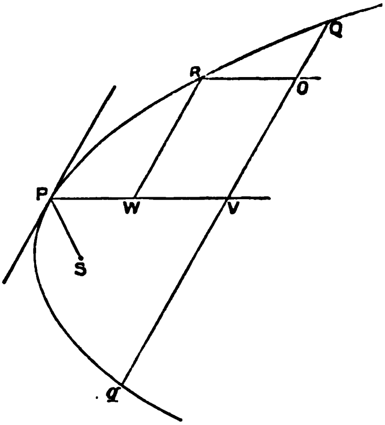
$$\therefore QW^2=4SR.PV.$$

PROPOSITION XXI.

If through any point O a diameter OR be drawn to meet the parabola at R, and also a chord QOq be drawn parallel to a fixed straight line, the rectangle OQ.Oq varies directly as OR.

Let the tangent at P be parallel to the fixed straight line.

Draw the diameter PV, bisecting Qq at V (Prop. 8), and the ordinate RW.



$$\begin{aligned} \text{Then } OQ.Oq &= VQ^2 - VO^2 \text{ (Euc. II. 5, Cor.)} \\ &= VQ^2 - RW^2 \\ &= 4SP.PV - 4SP.PW \text{ (Prop. 9)} \\ &= 4SP.(PV - PW) = 4SP.RO; \end{aligned}$$

$\therefore\ OQ.Oq \propto RO$, since P is a fixed point.

PROPOSITION XXII.

If through any point O two chords QOq, Q′Oq′ be drawn parallel to two fixed straight lines, the ratio of the rectangles OQ.Oq, OQ′.Oq′ is invariable.

Draw the diameter OR, and let the tangents at P and P' be parallel to the two fixed lines respectively, and therefore to Qq

and $Q'q'$. Then $OQ.Oq=4SP.OR$, and $OQ'.Oq'=4SP'.OR$ (Prop. 21).

$$\therefore \frac{OQ.Oq}{OQ'.Oq'}=\frac{SP}{SP'} \text{ which is constant for all positions of } O.$$

Cor. The point O may either be within or without the parabola, and consequently, if T be the point where the tangents at P and P' meet, $\frac{TP.TP}{TP'.TP'}$ or $\frac{TP^2}{TP'^2}=\frac{SP}{SP'}$.

Proposition XXIII.

If a circle be drawn cutting the parabola at four points A, B, C, D, then the straight lines AB, CD shall be equally inclined to the axis of the parabola.

Let AB, CD, produced if necessary, meet at E, and let TP, TQ be the two tangents to the parabola parallel to AB and CD respectively.

Then $\frac{EA.EB}{EC.ED}=\frac{SP}{SQ}=\frac{TP^2}{TQ^2}$ (Prop. 22).

But $EA.EB=EC.ED$ (Euc. III. 35 or 36, Cor.),

$\therefore$ $SP=SQ$ and $TP=TQ$.

Consequently SP and SQ are equally inclined to the axis; $\therefore$ TP and TQ and also AB and CD which are parallel to them, are equally inclined to the axis.

Similarly it might be proved that AC and BD are equally inclined to the axis, and also AD and BC.

Cor. Hence it follows, that if a circle touch a parabola at a point P and cut it at Q, R, the chord QR and the tangent at P are equally inclined to the axis.

MISCELLANEOUS PROBLEMS ON THE PARABOLA.

1. If from any point P of a circle, PC be drawn to the centre C, and a chord PQ drawn parallel to the diameter ACB be bisected at R, show that the locus of the intersection of CP and AR is a parabola.

2. PSp is a focal chord of a parabola, RXr the directrix meeting the axis at X; Q is any point on the curve; prove that if QP, Qp produced meet the directrix at R, r, half the latus rectum will be a mean proportional between XR, Xr.

3. If from the point of intersection of a pair of tangents, perpendiculars be drawn to the focal distances of the points of contact, these perpendiculars are equal to each other.

4. Two parabolas have a common focus S, and their axes in opposite directions; show that if a circle be drawn through the focus touching them at P and Q, the angle PSQ is two-thirds of a right angle.

5. If from the extremities of any double ordinate, chords be drawn through any point of a parabola, show that these chords intersect the axis at points equidistant from the vertex.

6. If from the vertex A, of a parabola any pair of chords AP, AQ be drawn at right angles to each other, prove that the chord PQ will pass through a fixed point on the axis.

7. Two parabolas have a common axis and vertex, and their concavities turned in opposite directions, the latus rectum of one being eight times that of the other; prove that the portion of the tangent to the former, intercepted between the common tangent and axis, is bisected by the latter.

8. If from the foot of the normal at any point P, a perpendicular be drawn to SP and produced to meet the tangent at P at T, and TD be drawn perpendicular to the ordinate of P, then $TD =$ the semi-latus rectum.

9. If PQ be a focal chord, and PM be drawn perpendicular to the directrix, prove that MQ passes through the vertex of the parabola.

10. G is the foot of the normal at a point P of a parabola, Q

is the middle point of SG, and X is the foot of the directrix; prove that $QX^2-QP^2=4AS^2$.

11. If the diameter through a point P on a parabola meet the tangent at the vertex at Z, and the focal distance of Z meet the normal at P at R; P and R will be equidistant from the tangent at the vertex.

12. If SY be the perpendicular from the focus on any tangent to a parabola, and Q be a point in SY, produced if necessary, such that $SY.SQ$ is constant, show that the locus of Q is a circle.

13. Given that PSQ is a focal chord of a parabola, of which S is the focus, determine the vertex by a geometrical construction.

14. If PG be the normal at a point P of a parabola, and upon PG, on the side remote from A, be described a segment of a circle containing an angle equal to half the angle which PG makes with the directrix, and this circle meet the axis again at L, and its tangent at P meet the axis at K, prove that $KL=4AN$.

15. PSP' is a focal chord of a parabola, the tangents at the extremities of which meet at Z; and AM is the perpendicular upon it from the vertex, prove that the rectangle $SZ \cdot AM$ is constant.

16. PSp is a focal chord of a parabola, and upon PS and pS as diameters circles are described; prove that the length of either of their common tangents is a mean proportional between AS and Pp.

17. A circle is described on the latus rectum of a parabola as diameter, and through the focus is drawn a straight line meeting the circle and parabola at P, Q respectively; show that the tangents at P and Q intersect on the latus rectum, or else on a line parallel to it at a distance from it equal to the latus rectum.

18. Two parabolas have a common vertex A and axis; an ordinate NPQ meets them at P, Q; a tangent at P meets the other parabola at R, R'; AR, AR' meet the ordinate at L, M. Prove that QN is a mean proportional between NL and NM.

19. If from any point R on a fixed tangent to a parabola

the other tangent RQ be drawn, then the angle QRS is constant in magnitude.

20. A circle passing through the focus of a parabola cuts the parabola at two points: prove that the difference between four times the angle between the tangents to the parabola at those points and the angle between the tangents to the circle at the same points is a multiple of two right angles.

21. A parabola touches the three sides of the triangle ABC, and through A, B, C lines are drawn respectively at right angles to SA, SB, SC: show that these lines pass through one point.

22. Given two tangents to a parabola and the directrix, find the focus and the points of contact.

23. Prove that the tangents TP, TQ drawn from any point T to a parabola are as their distances from the focus.

24. Prove by means of Prop. 8 Cor. 3, and Prop. 19, that if with the angular points A, B, C of a triangle, circles be described passing through a point S in the circumference of the circle circumscribing ABC, these three circles have a common chord.

25. Prove that the ratio of the rectangles contained by the segments of two focal chords is equal to the ratio of the chords.

26. A tangent to a parabola at P being given, give a geometrical construction for drawing another tangent which shall make a given angle with it.

27. Having given the tangents at two points of a parabola, determine the position of the focus.

28. If a quadrilateral be inscribed in a parabola of which the opposite angles are together equal to two right angles, the middle points of the opposite sides are equidistant from the axis.

29. Describe a parabola which shall have its axis parallel to a given straight line, and shall have a given finite straight line for a chord which at one of its extremities shall be a normal to the curve.

30. A chord of a parabola is drawn parallel to a given straight line, and on this chord as diameter a circle is described; prove that the distance between the middle points of this chord, and

of the chord joining the other two points of intersection of the circle and parabola, will be of constant length.

31. If QQ' be a chord of a parabola, and from any point T on the tangent at Q the diameter TRW be drawn to meet the curve at R, and QQ' at W, prove that

$$\frac{TR}{RW}=\frac{QW}{Q'W}.$$

32. Through any two points Q, R on the parabola, diameters are drawn which meet the tangent at another point P, at O and T respectively, prove that

$$\frac{OQ}{TR}=\frac{OP^2}{TP^2}.$$

33. Through a point P on the parabola two chords PQ, PQ' are drawn and produced if necessary to meet a given diameter at K, K' respectively: if R, W be the points where this diameter meets the curve and the chord QQ' respectively, prove that

$$\frac{KR}{K'R}=\frac{QW}{Q'W}.$$

34. If the tangents at P and Q intersect at T, and a line MSN be drawn through the focus parallel to the chord PQ, and meeting TP, TQ at M and N, show that

$$\frac{TP^2}{TQ^2}=\frac{SM}{SN}.$$

35. P is any point on a parabola and PB a chord through P. A second chord QQ' is drawn parallel to PB meeting the diameter through P at X. If the ordinates QM, $Q'M'$, BN be drawn to this diameter, show that $PN=XM \backsim XM'$.

36. A circle is drawn which touches a parabola at the point P and cuts it at Q and R; show that the ordinate of P bisects the angle QPR.

37. Through a given point within a parabola draw a chord which shall be divided in a given ratio at that point.

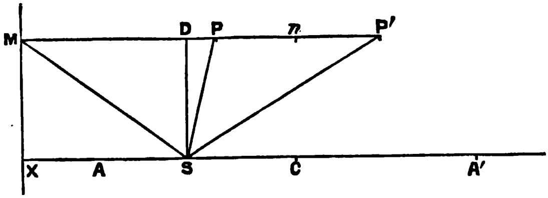
38. TP, TQ are tangents to a parabola at P and Q, and O is the centre of the circle circumscribing TPQ; show that TSO is a right angle.

CHAPTER III.

THE ELLIPSE.

Def. 1. An *Ellipse* is a plane curve traced out by a point, which moves in such a manner, that its distance from a fixed point bears a constant ratio, less than unity, to its perpendicular distance from a fixed straight line.

Def. 2. Let S be the focus, and MX the directrix (see Introduction). Draw SX perpendicular to MX, and divide SX at A, so that SA is to AX in the given ratio; also produce XS to A', so that SA' may be to $A'X$ in the given ratio: then the points A, A' which are on the ellipse are called its *Vertices*.



Note. If P be any point on the ellipse, and PM be drawn perpendicular to the directrix, then by definition

$$\frac{SP}{PM}=\frac{SA}{AX}=\frac{SA'}{A'X}=\text{the eccentricity.}$$

$$\text{Also } \frac{SP}{SA}=\frac{PM}{AX} \text{ or } \frac{SP}{SA'}=\frac{PM}{A'X}.$$

Def. 3. If AA' be bisected at C, the point C is called the *Centre* of the ellipse.

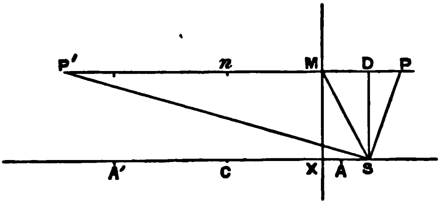
Def. 4. AA' is called the *Axis Major* or *Transverse Axis*.

CHAPTER IV.

THE HYPERBOLA.

Def. 1. A *Hyperbola* is a plane curve traced out by a point, which moves in such a manner, that its distance from a fixed point bears a constant ratio, greater than unity, to its perpendicular distance from a fixed straight line.

Def. 2. Let S be the focus and MX the directrix (see Introduction). Draw SX perpendicular to MX, and divide SX at A, so that SA is to AX in the given ratio; also produce SX to A', so that SA' may be to $A'X$ in the given ratio: then the points A, A' which are on the hyperbola are called its *Vertices*.



Note. If P be any point on the hyperbola, and PM be drawn perpendicular to the directrix, then by definition,

$$\frac{SP}{PM}=\frac{SA}{AX}=\frac{SA'}{A'X}=\text{the eccentricity.}$$

$$\text{Also }\frac{SP}{SA}=\frac{PM}{AX}\text{ or }\frac{SP}{SA'}=\frac{PM}{A'X}.$$

Def. 3. If AA' be bisected at C, the point C is called the *Centre* of the hyperbola.

Def. 4. AA' is called the *Transverse Axis*.

PROPOSITION I.

The Ellipse has two foci and directrices.

Let a straight line MPP' (see Fig. on page 30), drawn perpendicular to the directrix, meet the ellipse at P, P'. Draw SD perpendicular to PP', and join SP, SP', SM.

Then $SP'^2=SM^2+P'M^2-2P'M.MD$ (Euc. II. 13),

and $SP^2=SM^2+PM^2-2PM.MD$ (Euc. II. 13);

$$\therefore\ SP'^2-SP^2=P'M^2-PM^2-2MD.(P'M-PM),$$

$$\therefore\ \frac{SP'^2-SP^2}{P'M^2-PM}=1-\frac{2MD}{P'M+PM}. \qquad (1.)$$

$$\text{But}\quad \frac{SA^2}{AX^2}=\frac{SP'^2}{P'M^2}=\frac{SP^2}{PM^2}\ \text{(Def. 1)}$$

$$=\frac{SP'^2-SP^2}{P'M^2-PM^2};$$

$$\therefore\ \text{from (1.)}\ \frac{2MD}{P'M+PM}=1-\frac{SA^2}{AX^2}=\frac{AX^2-SA^2}{AX^2},$$

$$\text{or}\ \frac{2SX}{P'M+PM}=\frac{SX.(AX-SA)}{AX^2}\ \text{(Euc. II. 5, Cor.)};$$

$$\therefore\ P'M+PM=\frac{2AX^2}{AX-AS}. \qquad (2.)$$

Let PP' be bisected at n, then

$$P'M+PM=(nM+nP')+(nM-nP)$$
$$=2nM;$$

$$\therefore\ \text{from (2.)}\ nM=\frac{AX^2}{AX-AS}.$$

That is, the length of nM is constant for all positions of the the line PP', and hence $nM=CX$. Therefore Cn is parallel to MX or perpendicular to AA'; and n being the middle point of PP', it follows that the curve is symmetrical with regard to the line drawn through the centre C at right angles to the axis-major, and consequently there must be another focus and directrix at distances from C equal to CS and CX respectively.

By this proposition, and from the consideration that the eccen-

Proposition I.

The hyperbola has two foci and directrices.

Let a straight line PMP' (see Fig. on page 31), drawn perpendicular to the directrix, meet the hyperbola at P, P'. Draw SD perpendicular to PP', and join SP, SP', SM.

Then $SP'^2=SM^2+P'M^2+2P'M.MD$ (Euc. II. 12),

and $SP^2=SM^2+PM^2-2PM.MD$ (Euc. II. 13);

$$\therefore\ SP'^2-SP^2=P'M^2-PM^2+2MD.(P'M+PM),$$

$$\therefore\ \frac{SP'^2-SP^2}{P'M^2-PM^2}=1+\frac{2MD}{P'M-PM}. \qquad (1.)$$

But $$\frac{SA^2}{AX^2}=\frac{SP'^2}{P'M^2}=\frac{SP^2}{PM^2} \quad \text{(Def. 1)}$$

$$=\frac{SP'^2-SP^2}{P'M^2-PM^2};$$

$$\therefore \text{ from (1.) } \frac{2MD}{P'M-PM}=\frac{SA^2}{AX^2}-1=\frac{SA^2-AX^2}{AX^2},$$

$$\text{or } \frac{2SX}{P'M-PM}=\frac{SX.(SA-AX)}{AX^2} \quad \text{(Euc. II. 5, Cor.)};$$

$$\therefore\ P'M-PM=\frac{2AX^2}{SA-AX}. \qquad (2.)$$

Let PP' be bisected at n, then

$$P'M-PM=(P'n+nM)-(Pn-nM)$$
$$=2nM;$$

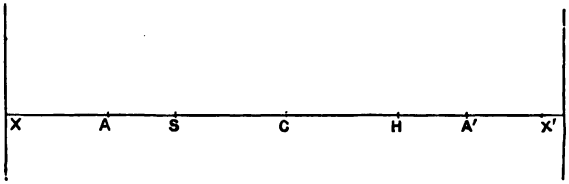
$$\therefore \text{ from (2.) } nM=\frac{AX^2}{SA-AX}.$$

That is, the length of nM is constant for all positions of the line PP', and hence $nM=CX$. Therefore Cn is parallel to MX or perpendicular to AA'; and n being the middle point of PP', it follows that the curve is symmetrical with regard to the line drawn through the centre C at right angles to the transverse axis, and consequently there must be another focus and directrix at distances from C equal to CS and CX respectively.

By this proposition, and from the consideration that the eccentricity is less than unity, it may be seen that the ellipse is a closed curve, which is divided into four equal and similar portions by ACA' and the line through C at right angles to AA'. The form will however be more apparent after we have proved Prop. 3.

PROPOSITION II.

If C be the centre of an ellipse, A a vertex, S a focus, and X the foot of the corresponding directrix, then each of the ratios $\frac{CS}{CA}$, $\frac{CA}{CX}$ is equal to the eccentricity.



Let A' and H be the other vertex and focus, and X' the foot of the other directrix.

$$\text{Then the eccentricity} = \frac{SA'}{A'X} = \frac{SA}{AX}$$

$$= \frac{SA'-SA}{A'X-AX} = \frac{SA'+SA}{A'X+AX}$$

$$= \frac{SH}{AA'} = \frac{AA'}{XX'} \text{ (since } SA = HA' \text{ and } AX = A'X')$$

$$= \frac{2CS}{2CA} = \frac{2CA}{2CX}$$

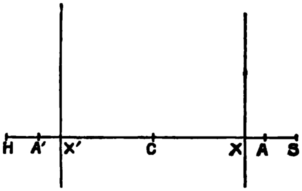
$$= \frac{CS}{CA} = \frac{CA}{CX}.$$

COR. 1. Hence it follows that CA is a mean proportional between CS and CX, or that $CS.CX = CA^2$.

tricity is greater than unity, it may be seen that the hyperbola consists of two similar branches diverging outwards from A and A'; and that no portion of the curve lies between A and A'. The form will however be more apparent after we have proved Prop. 3.

PROPOSITION II.

If C be the centre of a hyperbola, A a vertex, S a focus, and X the foot of the corresponding directrix, then each of the ratios $\frac{CS}{CA}$, $\frac{CA}{CX}$ *is equal to the eccentricity.*



Let A', and H be the other vertex and focus and X' the foot of the other directrix.

Then the eccentricity $=\frac{SA'}{A'X}=\frac{SA}{AX}$

$$=\frac{SA'+SA}{A'X+AX}=\frac{SA'-SA}{A'X-AX}$$

$$=\frac{SH}{AA'}=\frac{AA'}{XX'} \quad (\text{since } SA=HA' \text{ and } AX=A'X')$$

$$=\frac{2CS}{2CA}=\frac{2CA}{2CX}$$

$$=\frac{CS}{CA}=\frac{CA}{CX}.$$

COR. 1. Hence it follows that CA is a mean proportional between CS and CX, or that $CS.CX=CA^2$.

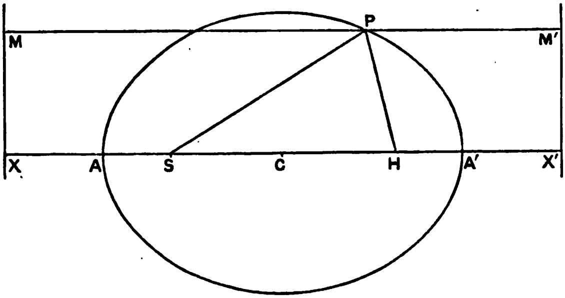
Cor. 2. Since $\frac{CS}{CA} = \frac{CA}{CX}$,

$$\therefore \frac{CS}{CX} = \frac{CS^2}{CA^2} \text{ (Def. of duplicate ratio).}$$

$$\text{Also } \frac{CS}{CX} = \frac{CA^2}{CX^2}.$$

Proposition III.

The sum of the focal distances of any point on the ellipse is constant and equal to the axis major.



Let P be any point on the ellipse. Join P to the two foci S and H, and draw MPM' perpendicular to the two directrices MX, $M'X'$. Then the eccentricity of the ellipse

$$= \frac{HP}{PM'} = \frac{SP}{PM} = \frac{HA}{AX'} = \frac{SA}{AX}$$

$$= \frac{HP+SP}{PM'+PM} = \frac{HA+SA}{AX'+AX}$$

$$= \frac{HP+SP}{MM'} = \frac{AA'}{XX'} \text{ (since } SA = HA'\text{);}$$

and therefore since $MM' = XX'$, we have

$$HP+SP = AA'.$$

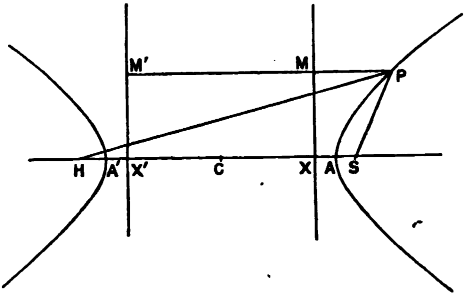
Cor. 2. Since $\frac{CS}{CA} = \frac{CA}{CX}$,

$$\therefore \frac{CS}{CX} = \frac{CS^2}{CA^2} \quad \text{(Def. of duplicate ratio).}$$

$$\text{Also } \frac{CS}{CX} = \frac{CA^2}{CX^2}.$$

Proposition III.

The difference of the focal distances of any point on the hyperbola is constant and equal to the transverse axis.



Let P be any point on the hyperbola. Join P to the two foci S and H, and draw PMM' perpendicular to the two directrices MX, $M'X'$.

Then the eccentricity of the hyperbola

$$= \frac{HP}{PM'} = \frac{SP}{PM} = \frac{HA}{AX'} = \frac{SA}{AX}$$

$$= \frac{HP-SP}{PM'-PM} = \frac{HA-SA}{AX'-AX}$$

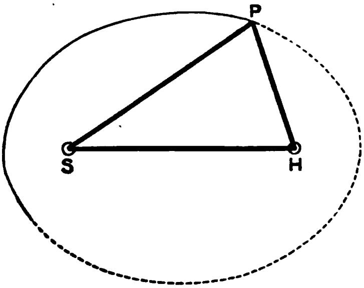
$$= \frac{HP-SP}{MM'} = \frac{AA'}{XX'} \text{ (since } SA=HA'\text{);}$$

and therefore since $MM'=XX'$, we have

$$HP-SP=AA'.$$

HOW TO DRAW AN ELLIPSE.

From the last proposition we obtain a simple method for constructing an ellipse. Let a string, the ends of which are tied together, be placed round two pins fixed in the plane of the paper at S and H. Then a pencil P, placed in contact with the string so as to keep it tight, will, if moved round S and H, trace out an ellipse of which S and H are the foci, and whose axis major is equal to the length of the portion of the string $SP+PH$.



The ellipse and hyperbola are sometimes defined from the property here used—the ellipse as a curve traced out by a point which moves in such a manner that the sum of its distances from two fixed points is constant; and the hyperbola as a curve traced out by a point which moves so that the difference of its distances from two fixed points is constant.

We will show in Chapter VI. how our definitions may be deduced from these.

EXAMPLES.

1. Prove that the focal distance of any point outside an ellipse bears to its distance from the directrix a ratio greater than the eccentricity of the ellipse.

2. Prove that the sum of the focal distances of any point outside an ellipse is greater than the axis major.

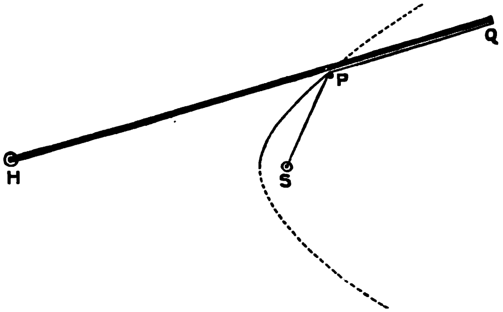
3. Show that *less* must be substituted for *greater* in these examples, for a point inside the ellipse.

4. Confocal ellipses cannot cut each other.

DEF. 5. The line BCB' drawn through the centre at right angles to the axis major and terminated by the ellipse is called the *Axis Minor* or *Conjugate Axis* of the ellipse.

How to Draw a Hyperbola.

From the last proposition we obtain a simple method for constructing a hyperbola. Let *HQ* be a straight rod capable of moving round *H* in the plane of the paper; and let *SPQ* be a string whose length is less than that of the rod, one of whose extremities is fastened to the end *Q* of the rod, and the other to a fixed point *S*, then a pencil placed at *P* so as to keep the string tight and the portion *PQ* in contact with the rod, will, as the rod turns about *H*, trace out a hyperbola whose foci are *S* and *H*, and whose transverse axis is equal to the difference between the lengths of the rod and string.



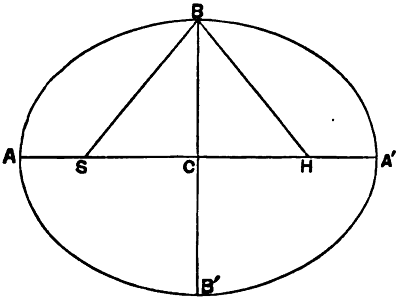
By this construction the right-hand branch of the curve is drawn. If the string be taken as much longer than the rod as it is now shorter, the other branch may be traced out.

Ex. Establish for the hyperbola propositions corresponding to those given in the four examples on the ellipse.

Def. 5. Draw *CB* through the centre at right angles to the transverse axis and of such length that $BC^2 = SC^2 - AC^2$ and produce *BC* to *B'* making $B'C = BC$, then *BCB'* is called the *Conjugate Axis* of the hyperbola.

PROPOSITION IV.

$BC^2 = AC^2 - SC^2 = AS.SA'$.



Join B with the foci S, H. Then $BS = BH$ (Euc. I. 4), and $BS + BH = AA'$ (Prop. 3); $\therefore BS = AC$.

But $BC^2 = BS^2 - SC^2$ (Euc. I. 47)

$= AC^2 - SC^2$

$= (AC - SC).(AC + SC)$

$= AS.SA'$ (since $AC = CA'$).

DEF. 6. Any line drawn through the centre, and terminated by the ellipse, is called a *Diameter*.

It is manifest, from the fact that the ellipse is symmetrical with respect to the two axes, that every diameter is bisected at the centre.

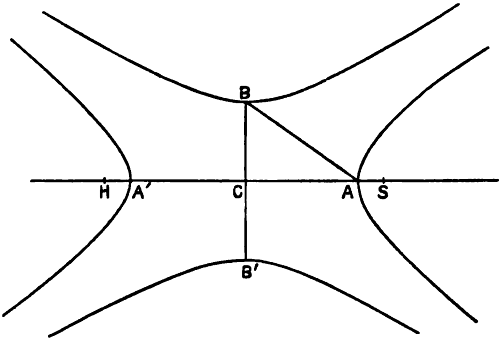
EXAMPLES.

Find the numerical values of the eccentricity, and of the ratio of the axes of the ellipse in each of the following cases :—

1. When the triangle SBH is equilateral.
2. When the triangle SBB' is equilateral.
3. When the triangle SBH is right-angled.
4. When the triangle SBH is similar to the one described in Euc. IV. 10.

PROPOSITION IV.

$BC^2 = AS.SA'$ *and* $AB = CS$.



$$
\begin{aligned}
BC^2 &= SC^2 - AC^2 \text{ (Def. 5)}\\
&= (SC-AC).(SC+AC)\\
&= AS.SA' \text{ (since } AC=CA'\text{)}.
\end{aligned}
$$

$$
\begin{aligned}
\text{Again } AB^2 &= BC^2 + AC^2 \text{ (Euc. I. 47)}\\
&= SC^2 - AC^2 + AC^2 \text{ (Def. 5)}\\
&= SC^2;\ \therefore\ AB = SC.
\end{aligned}
$$

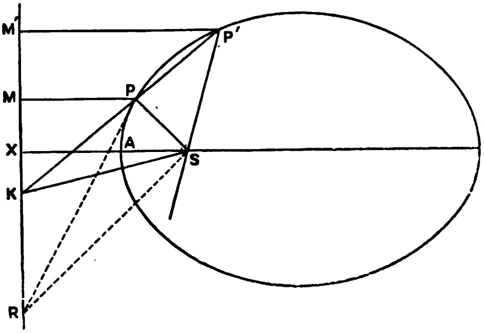
DEF. 6. The hyperbola whose transverse axis is BB' and conjugate axis AA', is called the *Conjugate Hyperbola:* its foci S', H' are in BB' produced at a distance from C equal to CS, since $CS' = CH' = AB = CS$.

DEF. 7. Any line drawn through the centre, and terminated by the hyperbola or its conjugate, is called a *Diameter* of the hyperbola.

It is manifest, from the fact that the hyperbola and its conjugate are symmetrical with respect to the two axes, that every diameter is bisected at the centre.

PROPOSITION V.

If the straight line joining two points P, P' on an ellipse, meet a directrix at K, the straight line joining K with the corresponding focus S will bisect the exterior angle between PS and $P'S$.



Draw PM, $P'M'$ perpendicular to the directrix.

$$\text{Then } \frac{SP}{SP'} = \frac{PM}{P'M'} \text{ (Def. 1)}$$

$$= \frac{PK}{P'K} \text{ (Euc. VI. 4. From triangles } PKM, P'KM').$$

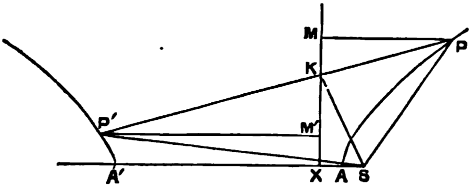
Consequently SK bisects the exterior angle of the triangle PSP' (Euc. VI. A).

COR. If the point P' moves along the curve up to P, $P'PK$ becomes ultimately the tangent at P (Def. 8, Introduction), and the interior angle between SP and SP' becomes indefinitely small, while the exterior angle approaches two right angles; and therefore the angle PSK, which is always half of this exterior angle, becomes ultimately a right angle. Hence if PR is the tangent, PSR is a right angle; that is, the portion of the tangent intercepted between the point of contact and the directrix subtends a right angle at the corresponding focus.

PROPOSITION V.

If the straight line joining two points P, P′ on a hyperbola meet a directrix at K, the straight line joining K with the corresponding focus will bisect the exterior or interior angle between PS and P′S according as P and P′ are on the same or different branches of the hyperbola.

First, if P and P' are on the same branch of the hyperbola, the proof is the same as for the ellipse.



Second, let P, P' be on different branches. Draw PM, $P'M'$ perpendicular to the directrix.

$$\text{Then } \frac{SP}{SP'} = \frac{PM}{P'M'} \quad \text{(Def. 1)}$$

$$= \frac{PK}{P'K} \quad \text{(Euc. VI. 4. From triangles } PKM,\ P'KM'\text{)}.$$

Consequently SK bisects the angle PSP' (Euc. VI. 3).

COR. As in the ellipse, if P and P' be supposed to be on the same branch, and if P' move up to and coincide with P, it may be shown that the portion of the tangent intercepted between the point of contact and the directrix subtends a right angle at the corresponding focus.

EXAMPLES.

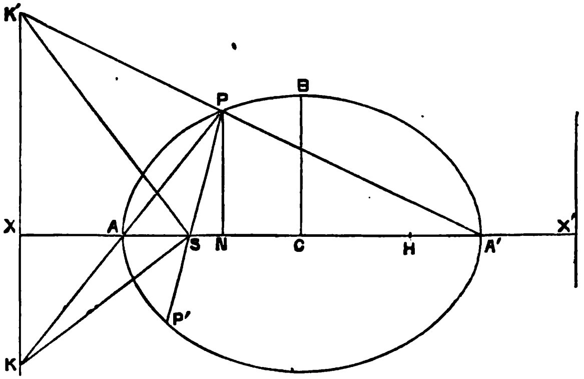
1. Show that the tangents at the extremities of any focal chord of an ellipse or hyperbola intersect on the corresponding directrix.

2. The tangents at the extremities of the latus rectum meet the axis at the point X.

PROPOSITION VI.

The square on the ordinate of any point on an ellipse is to the rectangle contained by the corresponding abscissæ as the square on the semi-axis minor is to the square on the semi-axis major.

$$\frac{PN^2}{AN.A'N} = \frac{BC^2}{AC^2}.$$



Join PA, PA', and produce them to meet one directrix at K, K' respectively; also join P, K, K' with the corresponding focus S, and produce PS to P'.

Then SK bisects the angle ASP' and SK' bisects ASP (Prop. 5);

∴ angle KSK' is half of ASP' and ASP together;

∴ KSK' is a right angle.

Again from the similar triangles APN and AKX we have

$$\frac{PN}{AN} = \frac{KX}{AX}. \qquad (1.)$$

Also from the similar triangles $A'PN$ and $A'K'X$ we have

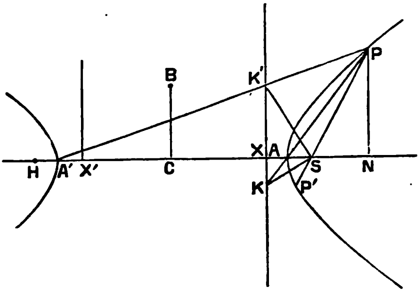
$$\frac{PN}{A'N} = \frac{K'X}{A'X}. \qquad (2\,)$$

Combining (1.) and (2.) we obtain

Proposition VI.

The square on the ordinate of any point on a hyperbola is to the rectangle contained by the corresponding abscissæ as the square on the semi-conjugate axis is to the square on the semi-transverse axis.

$$\frac{PN^2}{AN.A'N} = \frac{BC^2}{AC^2}.$$



Join PA, PA', and let them meet one directrix at K, K' respectively; also join P, K, K' with the corresponding focus S, and produce PS to P'.

Then SK bisects the angle ASP' and SK' bisects $A'SP$ (Prop. 5);

∴ angle KSK' is half ASP' and $A'SP$ together;

∴ KSK' is a right angle.

Again from the similar triangles APN and AKX we have

$$\frac{PN}{AN} = \frac{KX}{AX}. \qquad (1.)$$

Also from the similar triangles $A'PN$ and $A'K'X$ we have

$$\frac{PN}{A'N} = \frac{K'X}{A'X}. \qquad (2)$$

Combining (1.) and (2.) we obtain

$$\frac{PN^2}{AN.A'N}=\frac{KX.K'X}{AX.A'X}$$

$$=\frac{SX^2}{AX.A'X}\text{(Since } KSK' \text{ is a right angle, Euc. VI. 8, Cor.)}$$

And this last ratio is constant, consequently $\frac{PN^2}{AN.A'N}$ remains unaltered wherever P be taken.

Therefore we have

$$\frac{PN^2}{AN.A'N}=\frac{BC^2}{AC.A'C}$$

$$=\frac{BC^2}{AC^2}.$$

The proposition may be finished in the same manner as it is proved in the hyperbola.

Cor. If SL be the semi-latus rectum, we have from the proposition

$$\frac{SL^2}{AS.A'S}=\frac{BC^2}{AC^2},$$

$$\text{or } \frac{SL^2}{BC^2}=\frac{BC^2}{AC^2} \text{ (Prop. 4);}$$

$$\therefore \frac{SL}{BC}=\frac{BC}{AC},$$

$$\text{or semi-latus rectum } = \frac{BC^2}{AC}.$$

Examples.

1. Prove that in the ellipse $\frac{CN^2}{AC^2}+\frac{PN^2}{BC^2}=1.$

2. Prove that if Pn be drawn perpendicular to the axis minor $\frac{Pn^2}{Bn.B'n}=\frac{AC^2}{BC^2}.$

3. Show that the circle is a particular case of the ellipse. What then becomes of the foci and the directrices?

4. Assuming this proposition, enunciate and prove its converse, both for the ellipse and hyperbola.

$$\frac{PN^2}{AN.A'N}=\frac{KX.K'X}{AX.A'X}$$

$$=\frac{SX^2}{AX.A'X}\text{ (since }KSK'\text{ is a right angle, Euc. VI. 8, Cor.)}$$

$$=\frac{SA+AX}{AX}\times\frac{SA'-A'X}{A'X}$$

$$=\frac{CS+CA}{CA}\times\frac{CS-CA}{CA}\quad\text{(Prop. 2)}$$

$$=\frac{CS^2-CA^2}{CA^2}$$

$$=\frac{BC^2}{AC^2}\quad\text{(Def. 5).}$$

Cor. If SL be the semi-latus rectum, we have from the proposition

$$\frac{SL^2}{AS.A'S}=\frac{BC^2}{AC^2},$$

$$\text{or }\frac{SL^2}{BC^2}=\frac{BC^2}{AC^2}\quad\text{(Prop. 4);}$$

$$\therefore\ \frac{SL}{BC}=\frac{BC}{AC},$$

$$\text{or semi-latus rectum }=\frac{BC^2}{AC}.$$

Examples.

1. Prove that in the hyperbola $\frac{CN^2}{AC^2}-\frac{PN^2}{BC^2}=1$.

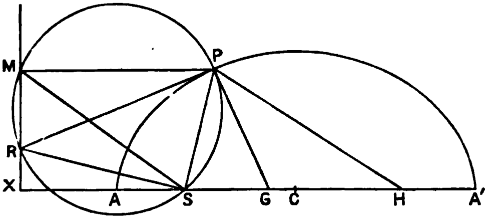
2. Is example 2 on the opposite page true for the hyperbola?

3. If the latus rectum be equal to one of the axes, then it is also equal to the other. What is the eccentricity of the hyperbola in this case?

4. If in the figure P be at the extremity of the latus rectum, prove that $KX=K'X$.

PROPOSITION VII.

If the normal at any point P on an ellipse meets the axis major at G, then the ratio of SG to SP is equal to the eccentricity: also the normal bisects the angle between the focal distances SP, HP.



Draw PM perpendicular to the directrix MX, and let the tangent at P meet the directrix at R. Join SR, SM.

Then since PMR and PSR are right angles (Prop. 5, Cor.), a circle described on PR as diameter will pass through M and S (Euc. III. 31), and PG, being at right angles to PR, is a tangent to the circle (Euc. III. 16); ∴ angle $SPG = SMP$ (Euc. III. 32). And since MP is parallel to SG, the angle $PSG = SPM$ (Euc. I. 29);

∴ The triangles SPG, PMS are equiangular (Euc. I. 32);

$$\therefore \frac{SG}{SP} = \frac{SP}{PM} \quad \text{(Euc. VI. 4)}$$

$$= \text{the eccentricity.}$$

Similarly it might be proved that

$$\frac{HG}{HP} = \text{the eccentricity.}$$

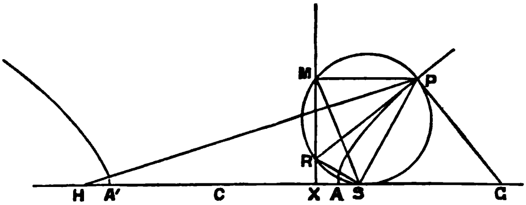
$$\therefore \frac{SG}{HG} = \frac{SP}{HP};$$

∴ PG bisects the angle SPH (Euc. VI. 3).

COR. 1. Since the tangent PR is at right angles to PG, it is manifest that PR bisects the angle between SP and HP produced; that is, the tangent at any point of an ellipse makes equal angles with the focal distances of that point.

Proposition VII.

If the normal at any point P on a hyperbola meets the transverse axis at G, then the ratio of SG to SP is equal to the eccentricity: also the normal makes equal angles with the focal distances SP, HP.



Draw *PM* perpendicular to the directrix *MX*, and let the tangent at *P* meet the directrix at *R*. Join *SR*, *SM*.

Then since *PMR* and *PSR* are right angles (Prop. 5, Cor.), a circle described on *PR* as diameter will pass through *M* and *S* (Euc. III. 31), and *PG*, being at right angles to *PR*, is a tangent to the circle (Euc. III. 16); ∴ angle *SPG*=*SMP* (Euc. III. 32). And since *MP* is parallel to *SG*, the angle *PSG*=*SPM* (Euc. I. 29);

∴ the triangles *SPG*, *PMS* are equiangular (Euc. I. 32);

$$\therefore \frac{SG}{SP}=\frac{SP}{PM} \quad \text{(Euc. VI. 4)}$$
$$=\text{the eccentricity.}$$

Similarly it might be proved that

$$\frac{HG}{HP}=\text{the eccentricity.}$$
$$\therefore \frac{SG}{HG}=\frac{SP}{HP};$$

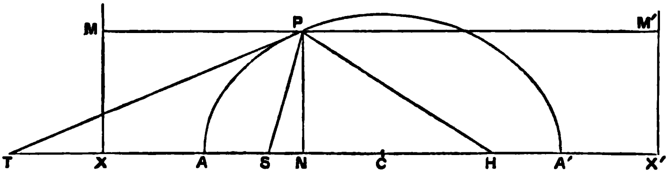
∴ *PG* bisects the exterior angle of the triangle *SPH* (Euc. VI. A).

Cor. 1. Since the tangent *PR* is at right angles to *PG*, it is manifest that *PR* bisects the angle *SPH*; that is, the tangent at any point of a hyperbola bisects the angle between the focal distances of that point.

COR. 2. From Cor. 1 it follows that the tangents at the vertices A, A' are at right angles to the axis major, and also that the tangents at the extremities of the axis minor are at right angles to that axis.

PROPOSITION VIII.

If the tangent and ordinate at a point P of the ellipse meet the axis major at T and N respectively, then $CN.CT=AC^2$.



Draw MPM' perpendicular to the directrices, and join SP, HP.

Then since PT bisects the angle between SP and HP produced (Prop. 7, Cor. 1),

$$\therefore \frac{HT}{ST}=\frac{HP}{SP} \quad \text{(Euc. VI. A.)}$$

$$=\frac{PM'}{PM} \quad \text{(Def. 1)}$$

$$=\frac{NX'}{NX} \quad \text{(Euc. I. 34).}$$

$$\therefore \frac{HT+ST}{HT-ST}=\frac{NX'+NX}{NX'-NX},$$

$$\text{or } \frac{(CT+CS)+(CT-CS)}{(CT+CS)-(CT-CS)}=\frac{(CX+CN)+(CX-CN)}{(CX+CN)-(CX-CN)},$$

$$\therefore \frac{2CT}{2CS}=\frac{2CX}{2CN};$$

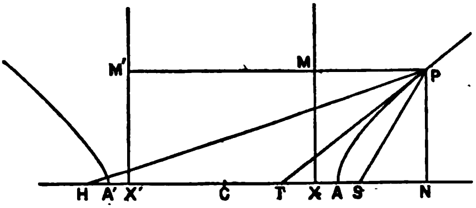
$$\therefore CT.CN=CX.CS$$

$$=AC^2 \text{ (Prop. 2, Cor. 1).}$$

Cor. 2. From Cor. 1 it follows that the tangents at the vertices A, A' are at right angles to the transverse axis. Also the tangents at the vertices B, B' of the conjugate hyperbola are at right angles to BB'.

Proposition VIII.

If the tangent and ordinate at a point P of the hyperbola meet the transverse axis at T and N respectively, then $CN.CT=AC^2$.



Draw PMM' perpendicular to the directrices, and join SP, HP.

Then since PT bisects the angle SPH (Prop. 7, Cor. 1),

$$\therefore \frac{HT}{ST}=\frac{HP}{SP} \quad \text{(Euc. VI. 3)}$$

$$=\frac{PM'}{PM} \quad \text{(Def. 1)}$$

$$=\frac{NX'}{NX} \quad \text{(Euc. I. 34).}$$

$$\therefore \frac{HT-ST}{HT+ST}=\frac{NX'-NX}{NX'+NX},$$

$$\text{or } \frac{(CS+CT)-(CS-CT)}{(CS+CT)+(CS-CT)}=\frac{(CN+CX)-(CN-CX)}{(CN+CX)+(CN-CX)}$$

$$\therefore \frac{2CT}{2CS}=\frac{2CX}{2CN};$$

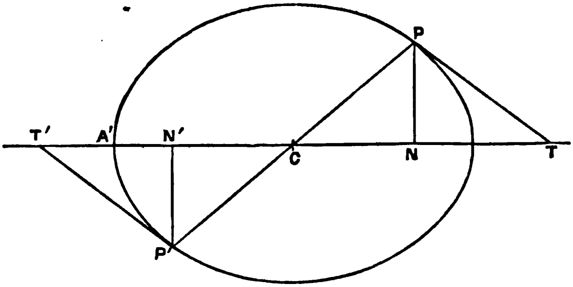
$$\therefore CT.CN=CX.CS$$

$$=AC^2 \quad \text{(Prop. 2, Cor. 1).}$$

PROPOSITION IX.

The tangents to an ellipse at the extremities of any diameter are parallel to each other.

Let PCP' be a diameter, and let the tangents at P, P' meet the axis major produced at T, T'. Draw the ordinates PN, $P'N'$.



Then since the curve is symmetrical with respect to the two axes, $CP=CP'$, $PN=P'N'$, $CN=CN'$.

Also $CN.CT=AC^2=CN'.CT'$ (Prop. 8);

$\therefore$ $CT=CT'$;

$\therefore$ The triangles CPT, $CP'T'$ are equal in all respects (Euc. I. 4), and angle $CTP=CT'P'$;

$\therefore$ PT is parallel to $P'T'$ (Euc. I. 27).

PROPOSITION X.

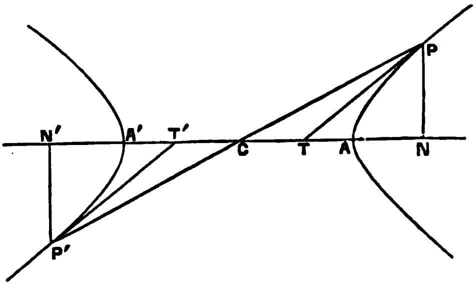
If to the tangent at any point P of an ellipse, perpendiculars SY, HZ be drawn from the foci, the feet of these perpendiculars Y, Z lie on the circumference of the circle described on the axis major as diameter; and the rectangle $SY.HZ=BC^2$.

Join SP, HP, CY, and produce SY, HP to meet at W. Then in the triangles SPY, WPY, since the angles at Y are right

Proposition IX.

The tangents to a hyperbola at the extremities of any diameter are parallel to each other.

Let PCP' be a diameter, and let the tangents at P, P' meet the transverse axis at T, T'. Draw the ordinates PN, $P'N'$.



Then since the curve is symmetrical with respect to the two axes, $CP=CP'$, $PN=P'N'$, $CN=CN'$.

Also $CN.CT=AC^2=CN'.CT'$ (Prop. 8);

$\therefore CT=CT'$;

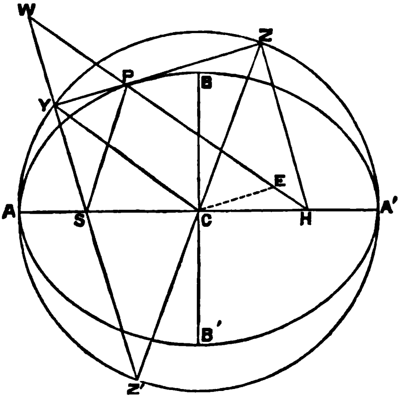
$\therefore$ The triangles CPT, $CP'T'$ are equal in all respects (Euc. I. 4), and angle $CTP=CT'P'$;

$\therefore PT$ is parallel to $P'T'$ (Euc. I. 27).

Proposition X.

If to the tangent at any point P of a hyperbola, perpendiculars SY, HZ be drawn from the foci, the feet of these perpendiculars Y, Z lie on the circumference of the circle described on the transverse axis as diameter; and the rectangle $SY.HZ=BC^2$.

Join SP, HP, CY, and let SY produced meet HP at W. Then in the triangles SPY, WPY, since the angles at Y are angles, and $SPY = WPY$ (Prop. 7, Cor. 1), and PY is common;



$\therefore$ $SY = WY$ and $SP = WP$;

$\therefore$ $HW = HP + SP = AA'$ (Prop. 3).

Again in the triangles SWH, SYC, since $SW = 2SY$ and $SH = 2SC$,

$\therefore$ HW is parallel to CY, and $HW = 2CY$ (Euc. VI. 2, 6, 4);

$\therefore$ $2CY = AA'$, $\therefore$ $CY = AC$;

$\therefore$ Y lies on the circle whose centre is C and radius AC; and similarly it may be shown that Z lies on the same circle.

Draw this circle and produce YS to meet it again at Z'. Then since ZYZ' is a right angle (hyp.),

$\therefore$ ZZ' is a diameter of the circle (Euc. III. 31), and consequently ZCZ' is a straight line.

Then in the triangles SCZ', HCZ, $SC = HC$, $CZ' = CZ$ and angle $SCZ' = HCZ$ (Euc. I. 15.), $\therefore$ $SZ' = HZ$ (Euc. I. 4);

$\therefore$ the rectangle $SY.HZ = SY.SZ'$

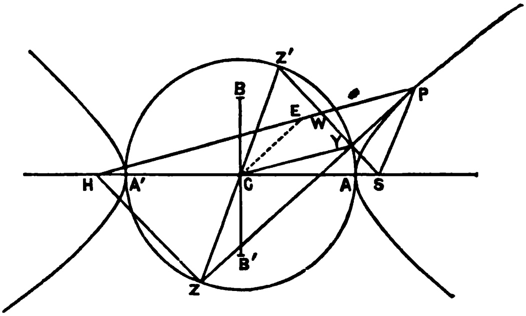
$= AS.SA'$ (Euc. III. 35)

$= BC^2$ (Prop. 4).

Cor. If CE be drawn parallel to the tangent PY to meet HP at E, then $CEPY$ is a parallelogram, and $\therefore$ $PE = CY = AC$.

Def. 7. The circle described on the axis major of an ellipse as diameter, is called the *Auxiliary Circle.*

right angles, and $SPY = WPY$ (Prop. 7, Cor. 1), and PY is common;



∴ $SY = WY$ and $SP = WP$;

∴ $HW = HP - SP = AA'$ (Prop. 3).

Again in the triangles SWH, SYC, since $SW = 2SY$ and $SH = 2SC$,

∴ HW is parallel to CY, and $HW = 2CY$ (Euc. VI. 2, 6, 4);

∴ $2CY = AA'$, ∴ $CY = AC$;

∴ Y lies on the circle whose centre is C and radius AC; and similarly it may be shown that Z lies on the same circle.

Draw this circle and produce SY to meet it again at Z'. Then since ZYZ' is a right angle (hyp.),

∴ ZZ' is a diameter of the circle (Euc. III. 31), and consequently ZCZ' is a straight line.

Then in the triangles SCZ', HCZ, $SC = HC$, $CZ' = CZ$ and angle $SCZ' = HCZ$ (Euc. I. 15), ∴ $SZ' = HZ$ (Euc. I. 4);

∴ the rectangle $SY.HZ = SY.SZ'$

$= SA.SA'$ (Euc. III. 36, Cor.)

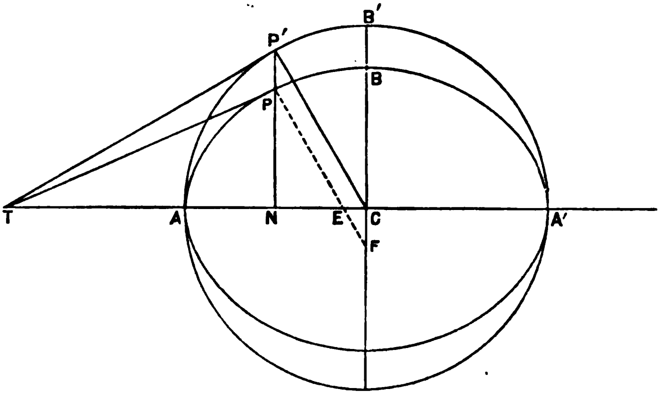
$= BC^2$ (Prop. 4).

Cor. If CE be drawn parallel to the tangent PY to meet HP at E, then $CEPY$ is a parallelogram, and ∴ $PE = CY = AC$.

Def. 8. The circle described on the transverse axis of a hyperbola as diameter, is called the *Auxiliary Circle.*

PROPOSITION XI.

If the ordinate NP of any point P on an ellipse be produced to meet the auxiliary circle at P′, then $\frac{PN}{P'N}=\frac{BC}{AC}$*: and the tangents at P and P′ meet the axis major produced at the same point.*



For $\frac{PN^2}{AN.A'N}=\frac{BC^2}{AC^2}$ (Prop. 6),

and $P'N^2=AN.A'N$ (Euc. III. 35);

$$\therefore \frac{PN^2}{P'N^2}=\frac{BC^2}{AC^2}, \text{ or } \frac{PN}{P'N}=\frac{BC}{AC}.$$

Again let the tangent at P meet the axis major produced at T. Join $P'T$ and CP'.

Then $CN.CT=AC^2$ (Prop. 8)

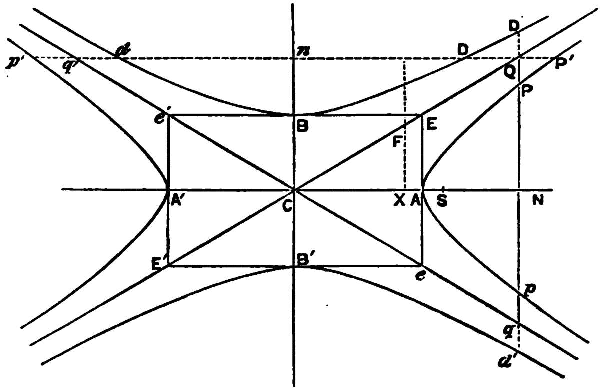
$=CP'^2$;

$$\therefore \frac{CN}{CP'}=\frac{CP'}{CT};$$

∴ The triangles CNP' and $CP'T$ are equiangular (Euc. VI. 6), and ∴ $CP'T$ is a right angle, and $P'T$ is the tangent to the circle at P' (Euc. III. 16).

Proposition XI.

If through any point Q on either of the diagonals of the rectangle $EeE'e'$, formed by drawing tangents at the vertices A, A', B, B' of a hyperbola and its conjugate, a line QN be drawn perpendicular to the transverse axis, meeting the hyperbola at P, p and the other diagonal at q, then $QP.Pq=BC^2$.



For $\quad \frac{QN^2}{AE^2}=\frac{CN^2}{AC^2}$ (From similar triangles),

or $\quad \frac{QN^2}{BC^2}=\frac{CN^2}{AC^2}$;

$$\therefore \frac{QN^2-BC^2}{BC^2}=\frac{CN^2-AC^2}{AC^2}$$

$$=\frac{AN.A'N}{AC^2} \quad \text{(Euc. II. 6)}$$

$$=\frac{PN^2}{BC^2} \quad \text{(Prop. 6)}.$$

$\therefore\ QN^2-BC^2=PN^2$,

$\therefore\ QN^2-PN^2=BC^2$,

or $\quad QP.Pq=BC^2$ (Euc. II. 5).

Now as Qq is removed further from A, Pq continually increases, and therefore QP must continually diminish, since their

Cor. If through P a line be drawn parallel to CP' meeting the axis at E, F, then $PF=CP'=AC$; and $\frac{PE}{P'C}=\frac{PN}{P'N}=\frac{BC}{AC}$; $\therefore$ $PE=BC$. Thus if PEF be a straight rod, and two fixed points E, F on it move along a pair of straight lines at right angles, then any point P on the rod will trace out an ellipse.

ON PROJECTIONS.

Def. 8. The foot of the perpendicular drawn from a point A to a plane P, is called the ***Projection*** of that point on the plane.

Def. 9. The projection of any line ABC on the plane P, is the locus of the projections of every point of the line on the plane.

Lemma 1. *The projection of a straight line AB on a plane P, is a straight line.*

Let a be the projection of A on the plane P. Draw the plane passing through AB and Aa; then all the lines drawn from points in AB perpendicular to the line of intersection of this latter plane and the plane P are parallel to Aa, and therefore perpendicular to P (Euc. XI. 8). Consequently the straight line of intersection of the two planes is the projection of AB.

Lemma 2. *The projection on a given plane of the tangent to a curve at any point is the tangent to the projection of the curve at the projection of the point.*

For if p, q be the projections of two points P, Q on the curve, the straight line pq is the projection of the secant PQ; and if Q move along the curve so as ultimately to coincide with P, then q will move along the projection so as ultimately to coincide with p; and as the secant PQ becomes ultimately the tangent at P, so will the secant pq become ultimately the tangent at p.

Lemma 3. *The projection of any figure on a plane P will be identical with its projection on any other plane which is parallel to P.*

This is obvious from the consideration that the perpendiculars drawn from the different points of the figure to the plane P remain the same for any parallel plane.

rectangle is constant; and by increasing Pq sufficiently, QP though it never can vanish, may be made less than any assignable quantity; hence the line CE gradually approaches the hyperbola, but never cuts it; and so for Ce. From this circumstance these lines are called the *Asymptotes* of the hyperbola.

Cor. 1. If Qn be drawn perpendicular to the conjugate axis, meeting the conjugate hyperbola at D, d, and the other asymptote at q', it may be proved in like manner that $QD.Dq'=AC^2$; and hence that CE, Ce' are likewise asymptotes of the conjugate hyperbola.

Cor. 2. It may also be proved in a similar manner that if Qn produced meet the original hyperbola at P', p', and QN produced meet the conjugate hyperbola at D', d', that

$$QP'.P'q'=AC^2, \text{ and } QD'.D'q=BC^2.$$

$$\text{Also} \quad Qp.pq=BC^2=Qd'.d'q,$$

$$\text{and } Qp'.p'q'=AC^2=Qd.dq'.$$

Cor. 3. Hence we see that $PQ=pq$, $P'Q=p'q'$, $DQ=dq'$ and $D'Q=d'q$.

Proposition XII.

The portion of an asymptote intercepted between the two directrices is equal to the transverse axis.

Let the asymptote CE meet the directrix XF (fig. of last Prop.) at F; then since FX, AE are parallel,

$$\frac{CF}{CE}=\frac{CX}{CA} \quad \text{(Euc. VI. 2)}$$

$$=\frac{CA}{CS} \quad \text{(Prop. 3).}$$

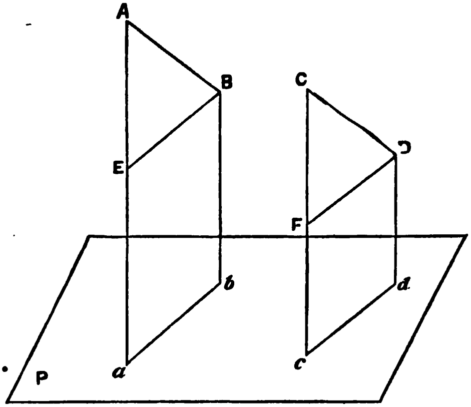
But $CE=AB=CS$ (Prop. 4);

$\therefore$ $CF=CA$, which proves the proposition, since CF is half of the portion of the asymptote intercepted between the directrices.

Proposition XII.

The projections of finite parallel straight lines on a given plane P, are parallel, and proportional to the given straight lines.

Let ab and cd be the projections on the plane P of the parallel straight lines AB and CD, then ab is parallel to cd, and $\frac{ab}{cd}=\frac{AB}{CD}$.



Since AB and Aa are respectively parallel to CD and Cc, the planes BAa and DCc are parallel (Euc. XI. 15), and ∴ ab and cd the intersections of these planes with P are parallel (Euc. XI. 16).

Draw BE, DF parallel to ab or cd. Then the sides of the triangles ABE, CDF are respectively parallel to each other, and consequently the triangles are equiangular (Euc. XI. 10);

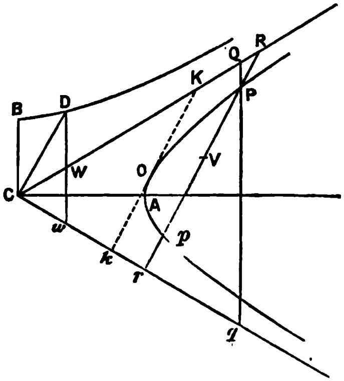
$$\therefore \frac{AB}{BE}=\frac{CD}{DF} \text{ (Euc. VI. 4);}$$

$$\therefore \frac{AB}{ab}=\frac{CD}{cd} \text{ (Euc. I. 34); or } \frac{AB}{CD}=\frac{ab}{cd}.$$

Proposition XIII.

If through any point P on a hyperbola any chord Pp be drawn meeting the asymptotes at R, r; and a line through the centre C be drawn parallel to this chord meeting the conjugate hyperbola at D, then $PR.Pr=CD^2$ and $PR=pr$.

Draw perpendiculars from P and D to the transverse axis meeting the asymptotes at Q, q, W, w, respectively; and bisect Rr at V.



Then the triangles DWC, PQR are equiangular, since their sides are respectively parallel to each other;

$$\therefore \frac{CD}{DW}=\frac{PR}{PQ} \quad \text{(Euc. VI. 4).}$$

Similarly, $\dfrac{CD}{Dw}=\dfrac{Pr}{Pq}$;

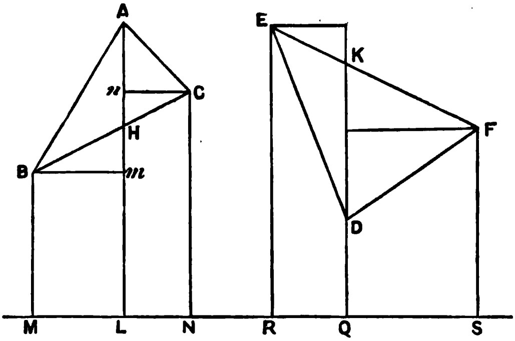
$$\therefore \frac{CD^2}{DW.Dw}=\frac{PR.Pr}{PQ.Pq}$$

But $DW.Dw=BC^2=PQ.Pq$ (Prop. 11, Cor. 2);

$$\therefore CD^2=PR.Pr$$
$$=VR^2-VP^2 \quad \text{(Euc. II. 5).}$$

Proposition XIII.

If triangles in the same plane be projected on another given plane P, the areas of the projections are proportional to the given triangles.



Let the plane of the paper contain the triangles ABC, DEF, and let MS be the intersection of this plane with the given plane P. Draw AL, BM, CN, DQ, ER, FS perpendicular to MS, and let AL, DQ meet BC, EF, at H, K respectively. Draw Bm, Cn perpendicular to AL.

Then area $ABC = ABH + ACH = \frac{1}{2}Bm.AH + \frac{1}{2}Cn.AH$
$= \frac{1}{2}MN.AH.$

Similarly it may be shown that $DEF = \frac{1}{2}RS.DK$.

And if abc, def be the projections of ABC, DEF, and h, k those of H, K, it may be shown in like manner that $abc = \frac{1}{2}MN.ah$, and $def = \frac{1}{2}RS.dk$;

$$\therefore \frac{ABC}{abc} = \frac{AH}{ah} = \frac{DK}{dk} \text{ (Prop. 12)} = \frac{DEF}{def}.$$

Cor. The proposition may be extended to any rectilineal figures, for these may always be divided into triangles. Hence it may be still further extended to any curvilinear figures.

Similarly it may be shown that

$$CD^2 = pR.pr = Vr^2 - Vp^2 = VR^2 - Vp^2;$$
$$\therefore VR^2 - VP^2 = VR^2 - Vp^2;$$
$$\therefore VP = Vp,$$

and consequently $PR = pr$.

Cor. 1. This proposition being true for all positions of the chord Pp, will remain true in the limiting position when the chord becomes a tangent; that is, if KOk be the tangent parallel to CD, we must have $OK.Ok = CD^2$, and $OK = Ok$; and consequently $OK = CD$.

Cor. 2. In Cor. 1 we have proved that the portion of the tangent at any point of a hyperbola, intercepted by the asymptotes, is bisected at the point of contact: and since no other line through O terminated by the asymptotes can be bisected at O, it follows that if a straight line drawn through any point of a hyperbola and terminated by the asymptotes be bisected at that point, the line is a tangent at that point.

Examples.

1. If from the foot of the ordinate PN of any point P on a hyperbola, a tangent NQ be drawn to the auxiliary circle, then $\frac{PN}{NQ} = \frac{BC}{AC}$.

2. Show that the feet of the perpendiculars drawn from the foci to the asymptotes lie on the directrices. Hence show, by aid of Prop. 5, Cor., that the asymptotes may be regarded as tangents to the hyperbola at points lying at an infinite distance.

THE RECTANGULAR HYPERBOLA.

Def. 9. If the axes of a hyperbola are equal, it is called a *Rectangular* or *Equilateral Hyperbola*.

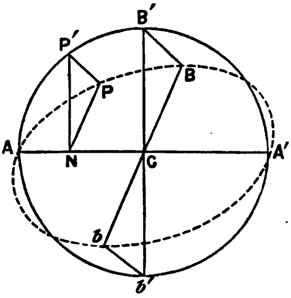
The former name comes from the obvious fact that the asymptotes are at right angles to each other.

PROPOSITION XIV.

The projection of a circle on a given plane is an ellipse.

Whatever may be the plane of projection, we can always suppose that it passes through the centre of the circle, since by Lemma 3, the projections on all parallel planes are the same.

Let then ACA' be the diameter in which the plane of projection cuts the given circle.



Draw the diameter $B'Cb'$ at right angles to ACA', and also draw in the plane of projection BCb at right angles to ACA'; then if $B'B$, $b'b$ be drawn perpendicular to BCb, B and b are the projections of B' and b' (Euc. XI. 11).

Similarly if P' be any point on the circle, and $P'N$ be drawn perpendicular to ACA' and NP at right angles to AA' in the plane of projection, and $P'P$ perpendicular to PN, then P is the projection of P'.

Then since the sides of the triangle $B'CB$ are respectively parallel to those of $P'NP$, the triangles are equiangular (Euc. XI. 10);

$$\therefore \frac{PN^2}{P'N^2}=\frac{BC^2}{B'C^2} \quad \text{(Euc. VI. 4).}$$

But $P'N^2=AN.A'N$ (Euc. III. 35), and $B'C=AC$,

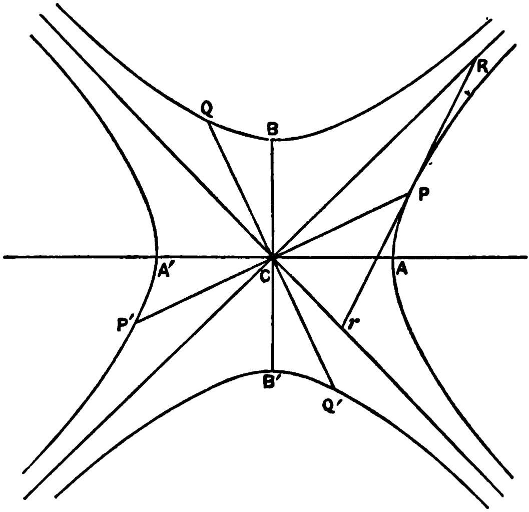
$$\therefore \frac{PN^2}{AN.A'N}=\frac{BC^2}{AC^2};$$

and therefore P lies on the ellipse whose semi-axes are AC and BC (Prop. 6), which proves the proposition, since P is the projection of any point P' on the given circle.

COR. 1. The centre of the circle projects into the centre of the ellipse, and every diameter of the former into a diameter of the latter.

Proposition XIV.

Any diameter of a rectangular hyperbola is equal to the portion of the tangent at its extremity which is intercepted by the asymptotes; and diameters at right angles to each other are equal.



Let PCP' be a diameter and RPr the tangent at its extremity P. Then since RCr is a right angle, the circle described on Rr as diameter will pass through C, and since Rr is bisected at P (Prop. 13, Cor. 1), P is the centre of this circle, and

$$\therefore\ CP=PR=Pr,$$
$$\therefore\ PP'=Rr.$$

Again let QCQ' be a diameter at right angles to CP, then since BCA is a right angle, $\therefore$ $QCB=PCA$; and as it is evident that the hyperbola and its conjugate are equal in all respects, it follows that diameters which make equal angles with their respective transverse axes must be equal, and $\therefore$ $QCQ'=PCP'$.

Cor. 2. It must be noticed that by suitably choosing the circle and the plane of projection, a circle may be projected into any ellipse whatever, for

$$BB'^2 = B'C^2 - BC^2 \text{ (Euc. I. 47)}$$
$$= AC^2 - BC^2 = CS^2 \text{ (Prop. 4)};$$
$$\therefore \frac{B'B}{B'C} = \frac{CS}{AC} = \text{ the eccentricity.}$$

And as BB' may have any magnitude not greater than $B'C$, it follows that the circle may be projected into an ellipse of any eccentricity; and by suitably choosing the diameter of the circle which is equal to the major axis of the ellipse, the latter may have any size.

Cor. 3. If the circle be turned about its diameter AA' until its plane coincides with the plane of the ellipse, then it becomes the auxiliary circle to the ellipse, and P' will lie in the ordinate NP produced.

Examples.

1. Prove that the projection of a parabola on a plane passing through its axis is another parabola.
2. Prove that the ratio of any plane area to its projection on any other plane, is equal to the secant of the angle between the two planes.
3. Find the area of an ellipse in terms of its axes.
4. The circle described about the triangle SPH (see Prop. 7 for ellipse or hyperbola), meets the normal and tangent at P on the conjugate axis.

CONJUGATE DIAMETERS.

Def. 10. If PCp be a diameter of an ellipse, and another diameter DCd be drawn parallel to the tangents at the extremities of the former, then DCd is said to be *Conjugate* to PCp.

Cor. Since $PR=CP$, $\therefore$ angle $PRC=PCR$; that is, the tangent and diameter at any point make equal angles with the asymptotes.

Examples.

1. Show that any line drawn parallel to an asymptote meets the hyperbola at one point only.

2. What is the eccentricity of the rectangular hyperbola?

3. In the rectangular hyperbola CS is bisected at X.

4. In the rectangular hyperbola $CN^2-PN^2=AC^2$.

5. In the rectangular hyperbola prove that the tangent at any point P makes the same angle with one axis that CP makes with the other.

6. Show that the normal PG at any point on the rectangular hyperbola is equal to CP.

7. The circle described about the triangle CRr (last Fig.) meets the normal at P on the two axes.

8. Show that the ordinate of any point on the rectangular hyperbola is a mean proportional between the abscissa CN and the sub-tangent; and that the sub-normal is equal to the abscissa.

9. If the hyperbola in the figure to Prop. 7 be rectangular, show that $SG=2PM$.

10. Prove that the projection of a rectangular hyperbola on any plane passing through either axis is a hyperbola, whose eccentricity is less or greater than that of the rectangular hyperbola according as the plane of projection passes through the transverse or conjugate axis. Hence show that a rectangular hyperbola can be projected into any other hyperbola.

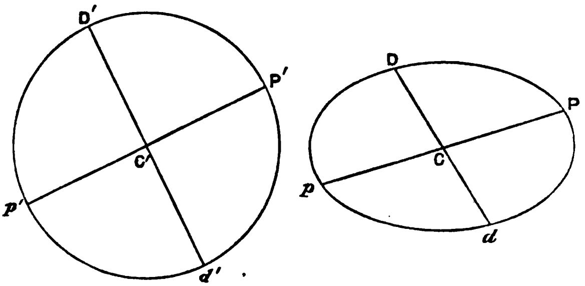
CONJUGATE DIAMETERS.

Def. 10. If PCp be a diameter of a hyperbola, and another diameter be drawn parallel to the tangents at the extremities of the former, meeting the conjugate hyperbola at D, d, then DCd is said to be *Conjugate* to PCp.

Proposition XV.

If DCd be a diameter conjugate to PCp, then PCp shall be conjugate to DCd.

Let $P'C'p'$ be a diameter of the circle which projects into PCp in the ellipse. Draw $D'C'd'$ parallel to the tangent to the circle at P', then $C'D'$ is at right angles to $C'P'$ (Euc. III. 18), and therefore $C'P'$ is parallel to the tangent at D'. Now the tangent at P' projects into the tangent at P (Lemma 2), and consequently $C'D'$ which is parallel to the tangent at P' projects into CD which is parallel to the tangent at P (Prop. 12). Again $C'P'$ and the tangent at D' project into CP and the tangent at D, and since the former are parallel, the latter must also be parallel, and hence CP is conjugate to CD.



Cor. It is seen from this, that diameters at right angles in a circle project into conjugate diameters of the ellipse.

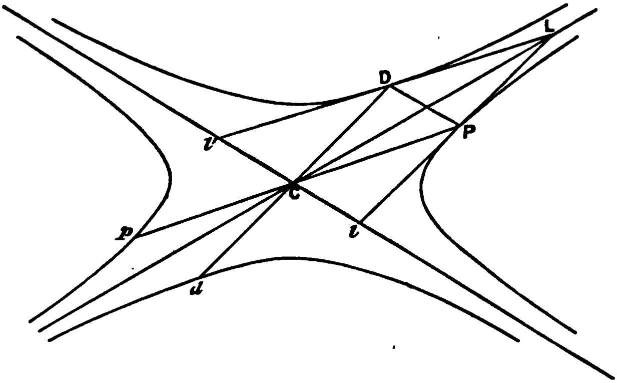
Examples.

1. What is the condition that $C'P'$, $C'D'$ may project into equal conjugate diameters?

2. The portions of the diagonals of the rectangle, formed by drawing tangents to an ellipse at the extremities of the axes, intercepted by the curve, are equiconjugate diameters.

PROPOSITION XV.

If DCd be a diameter conjugate to PCp, then PCp shall be conjugate to DCd.



Join PD, and let the tangent at P meet the asymptotes at L, l. Join LD and produce it to meet the other asymptote at l'.

Since CD is equal and parallel to PL (Prop. 13, Cor. 1),

$\therefore$ $CPLD$ is a parallelogram and $CP=DL$ (Euc. I. 33).

Similarly since CD is equal and parallel to Pl,

$\therefore$ $CDPl$ is a parallelogram and $DP=Cl$;

$\therefore$ lCl' and LDl' are parallel to DP and CP respectively,

$\therefore$ $CPDl'$ is a parallelogram and $Dl'=CP$.

But $DL=CP$, $\therefore$ $Dl'=DL$,

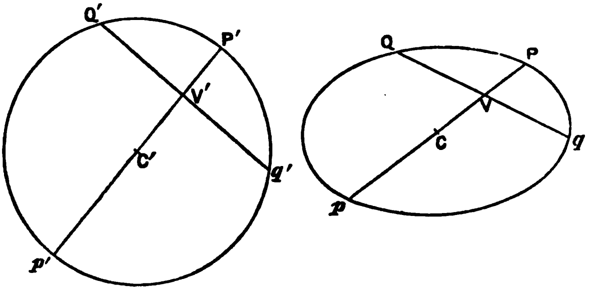
$\therefore$ LDl' is the tangent at D (Prop. 13, Cor. 2).

And CP is parallel to LDl' and is therefore conjugate to CD.

COR. Since the diagonals of a parallelogram bisect each other, it follows from the proposition that the line PD, joining the extremities of a pair of conjugate diameters, is parallel to one asymptote, lCl' and bisected by the other CL. We have also proved that $Cl=PD=Cl'$.

PROPOSITION XVI.

A diameter bisects all chords parallel to the tangents at its extremities.



Let $P'C'p'$ be a diameter of the circle which projects into PCp in the ellipse, and let $Q'V'q'$ be any chord parallel to the tangent to the circle at P', which projects into QVq, which is parallel to the tangent at P.

Then since $Q'V'q'$ is parallel to the tangent at P', it is at right angles to $C'P'$ (Euc. III. 18), and is therefore bisected by $C'P'$ at V' (Euc. III. 3).

But $\frac{QV}{qV}=\frac{Q'V'}{q'V'}$ (Prop. 12),

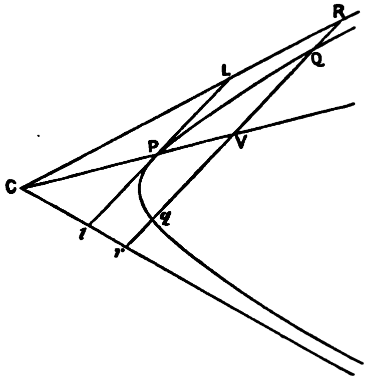
$\therefore$ $QV=qV$, which proves the proposition.

COR. Hence we see, both in the case of the ellipse and hyperbola, that the locus of the middle points of a series of parallel chords is a straight line. The same being true for the parabola (Chap. I., Prop. 8, Cor. 1), it follows that a diameter of the parabola really corresponds to a diameter of the ellipse or hyperbola.

DEF. 11. A line QV or qV parallel to the tangent at P is called an *Ordinate* to the diameter CP.

PROPOSITION XVI.

A diameter produced bisects all chords parallel to the tangents at its extremities.



Let Qq be any chord parallel to the tangent at P, and let CP produced cut the chord at V.

Produce the chord to meet the asymptotes at R, r, and let the tangent at P meet them at L, l.

Then by similar triangles

$$\frac{RV}{PL}=\frac{CV}{CP}=\frac{Vr}{Pl};$$

and $PL=Pl$ (Prop. 13, Cor. I),

$\therefore RV=Vr$.

Also $RQ=rq$ (Prop. 13),

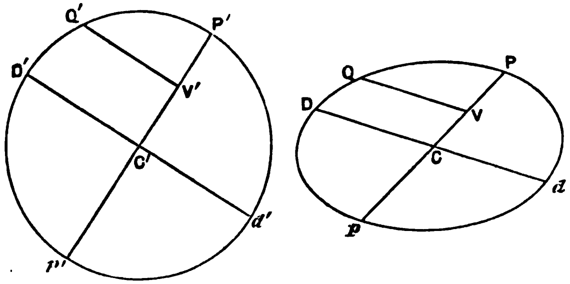
$\therefore VQ=Vq$, which proves the proposition.

DEF. 11. A line QV or qV parallel to the tangent at P is called an *Ordinate* to the diameter CP.

PROPOSITION XVII.

If PCp, DCd be a pair of conjugate diameters, and QV be an ordinate to PCp, then

$$\frac{PV.Vp}{CP^2}=\frac{QV^2}{CD^2}.$$



Let $P'C'p'$, $D'C'd'$ be two diameters at right angles in the circle, which project into the two conjugate diameters PCp, DCd in the ellipse (Prop. 15, Cor.); and let $Q'V'$ be an ordinate parallel to $C'D'$, which projects into QV.

Then $P'V'.V'p'=Q'V'^2$ (Euc. III. 35);

$$\therefore \frac{P'V'.V'p'}{C'P'^2}=\frac{Q'V'^2}{C'D'^2}.$$

But $\dfrac{P'V'}{C'P'}=\dfrac{PV}{CP}$ and $\dfrac{V'p'}{C'P'}=\dfrac{Vp}{CP}$ (Prop. 12),

$$\therefore \frac{P'V'.V'p'}{C'P'^2}=\frac{PV.Vp}{CP^2}.$$

Also $\dfrac{Q'V'^2}{C'D'^2}=\dfrac{QV^2}{CD^2}$ (Prop. 12);

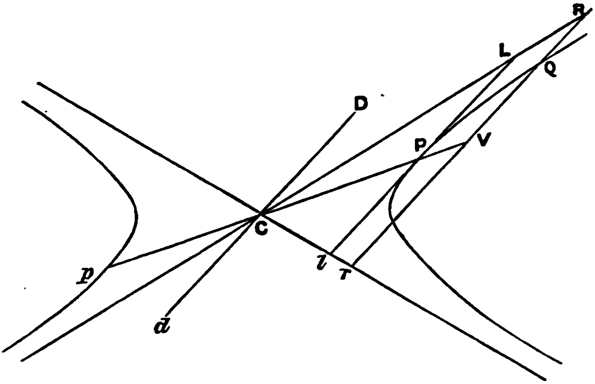
$$\therefore \frac{PV.Vp}{CP^2}=\frac{QV^2}{CD^2}.$$

Ex. If $CP=CD$, show that the circle, described about PQp, meets QV produced, on the ellipse.

PROPOSITION XVII.

If PCp, DCd be a pair of conjugate diameters, and QV be an ordinate to PCp, then

$$\frac{PV.Vp}{CP^2}=\frac{QV^2}{CD^2}.$$



Let the ordinate QV meet the asymptotes at R, r, and the tangent at P meet them at L, l.

Then $PL^2=CD^2$ (Prop. 13, Cor. 1)

$=RQ.Qr$ (Prop. 13)

$=VR^2-VQ^2$ (Euc. II. 5),

$\therefore QV^2=VR^2-PL^2$.

Again by similar triangles,

$$\frac{VR^2}{PL^2}=\frac{CV^2}{CP^2},$$

$$\therefore \frac{VR^2-PL^2}{PL^2}=\frac{CV^2-CP^2}{CP^2},$$

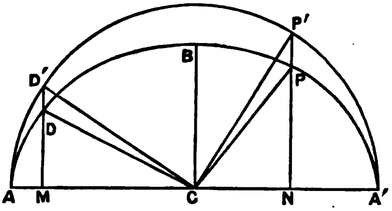
or $$\frac{QV^2}{CD^2}=\frac{CV^2-CP^2}{CP^2}$$

$$=\frac{PV.Vp}{CP^2} \quad \text{(Euc. II. 6)}.$$

PROPOSITION XVIII.

If CP, CD be two semi-conjugate diameters of an ellipse, then $CP^2+CD^2=AC^2+BC^2$.

Let the circle, which projects into the ellipse, be turned about its diameter till it coincides with the auxiliary circle of the ellipse, then the points P', D', which project into P and D, lie in the ordinates NP, MD produced (Prop. 14, Cor. 3), and CP' is at right angles to CD' (Prop. 15, Cor.).



Then since CMD' and $P'CD'$ are right angles, the angles $CD'M$ and $P'CN$ are each complementary to MCD', and therefore are equal, and since $CD'=CP'$, it follows that the triangles $CP'N$, $D'CM$ are equal in all respects (Euc. I. 26);
and $\therefore$ $CN=D'M$ and $P'N=CM$.

Also $CP^2+CD^2=CN^2+PN^2+CM^2+DM^2$ (Euc. I. 47)
$=CN^2+PN^2+P'N^2+DM^2$
$=CP'^2+PN^2+DM^2$ (Euc. I. 47)
$=AC^2+PN^2+DM^2$. (1.)

Again $\dfrac{PN^2}{BC^2}=\dfrac{P'N^2}{AC^2}$ and $\dfrac{DM^2}{BC^2}=\dfrac{D'M^2}{AC^2}$ (Prop. 11);

$$\therefore \frac{PN^2+DM^2}{BC^2}=\frac{P'N^2+D'M^2}{AC^2}=\frac{P'N^2+CN^2}{AC^2}$$

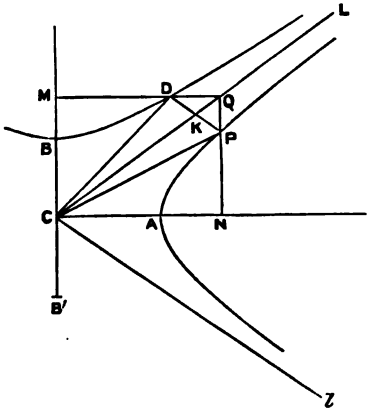
$$=\frac{CP'^2}{AC^2}=\frac{AC^2}{AC^2};$$

$\therefore$ $PN^2+DM^2=BC^2$.

Consequently from (1.) $CP^2+CD^2=AC^2+BC^2$.

Proposition XVIII.

If CP, CD be two semi-conjugate diameters of a hyperbola, then $CP^2-CD^2=AC^2-BC^2$.



Join PD meeting the asymptote CL at K, and let the ordinate PN meet the asymptote at Q.

Then since PD is parallel to Cl (Prop. 15, Cor.) and QP to $B'C$

$\therefore KPQ=B'Cl=BCL$

$=KQP$ (Euc. I. 29);

$\therefore KQ=KP$ (Euc. I. 6).

But $KP=DK$ (Prop. 15, Cor.), $\therefore KQ=DK$.

And it may be similarly shown that the ordinate DM meets the asymptote CL at a point whose distance from K is equal to DK or PK, that is at the point Q.

Now $CP^2-CD^2=CN^2+PN^2-CM^2-DM^2$ (Euc. I. 47)

$=QM^2+PN^2-QN^2-DM^2$

$=(QM^2-DM^2)-(QN^2-PN^2)$

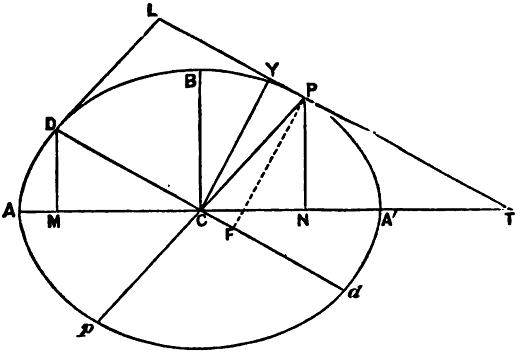
$=AC^2-BC^2$ (Prop. 11).

Cor. Since $CN = D'M$, $\therefore \frac{DM}{CN} = \frac{BC}{AC}$ (Prop. 11);

and also since $P'N = CM$, $\therefore \frac{PN}{CM} = \frac{BC}{AC}$.

Proposition XIX.

The parallelogram, formed by drawing tangents at the extremities of a pair of conjugate diameters, is of constant area, being equal to 4AC.BC.



Let CP, CD be conjugate, and let the tangents at P and D meet at L. Draw the ordinates PN, DM, and draw CY perpendicular to PL, and produce LP to meet the axis major produced at T. Then since YT is parallel to CD, the angle $YTC = MCD$, and therefore the triangles CYT, DMC are equiangular,

$$\therefore \frac{CY}{CT} = \frac{DM}{CD} \text{ (Euc. VI. 4);}$$

$$\therefore CY.CD = DM.CT.$$

$$\text{Again } \frac{DM}{CN} = \frac{BC}{AC} \text{ (Prop. 18, Cor.),}$$

$$\therefore \frac{DM.CT}{CN.CT} = \frac{BC}{AC},$$

Cor. It follows from what we have proved in the proposition, that if from any point Q in either asymptote ordinates QN, QM be drawn to both axes meeting the hyperbola and its conjugate at P, D respectively, then CP and CD are conjugate to each other.

Examples.

1. Show that the corollary on Prop. 18 for the ellipse is also true for the hyperbola.

2. Conjugate diameters of a rectangular hyperbola are equal to each other.

Proposition XIX.

The parallelogram, formed by drawing tangents to a hyperbola and its conjugate at the extremities of a pair of conjugate diameters, is of constant area, being equal to $4AC.BC$.

Let CP, CD be conjugate and $LPTl$, DL be the tangents at P and D, then $CPLD$ is one-fourth of the complete parallelogram, and since $Pl=PL$ (Prop. 13, Cor. 1), it follows that the triangle $CLl=$the parallelogram $CPLD$.

Draw the tangent at the vertex EAe and also draw PN, LM, lm perpendicular to the transverse axis. Then since P is the middle point of Ll, N must be the middle point of Mm and consequently $CM+Cm=2CN$.

Again from similar triangles we have

$$\frac{EA}{CA}=\frac{LM}{CM}=\frac{lm}{Cm}$$

$$=\frac{LM+lm}{CM+Cm}=\frac{LM+lm}{2CN};$$

$$\therefore LM+lm=2EA\times\frac{CN}{CA}=2BC\times\frac{CA}{CT} \quad \text{(Prop. 8)},$$

$$\therefore BC.CA=\tfrac{1}{2}CT.(LM+lm)$$

$$=\text{area } CTL+\text{area } CTl$$

$$=\text{area } CLl=\text{parallelogram } CPLD;$$

or $\frac{DM.CT}{AC^2}=\frac{BC}{AC}$ (Prop. 8);

$\therefore DM.CT=AC.BC=CY.CD.$

Now the parallelogram $CPLD$ is equal to the rectangle $CY.CD$ (Euc. I. 35), and $CPLD$ is obviously a quarter of the whole parallelogram formed by the tangents drawn at the extremities of the conjugate diameters, consequently this latter parallelogram is equal to $4AC.BC$.

The proposition can also be proved directly by the method of projections.

Cor. If PF be drawn perpendicular to CD, we have proved in the proposition, since $PF=CY$, that $PF.CD=AC.BC$.

Examples.

1. The diagonals of the complete parallelogram described in the proposition are conjugate to each other.

2. Show by means of Prop. 10 how to draw tangents to an ellipse or hyperbola from an external point.

3. Show that the chords, drawn from the extremities of a diameter to any point on the ellipse are parallel to a pair of conjugate diameters. (*N.B.* Such a pair of chords are called *Supplemental Chords.*)

4. If a circle be described on any diameter of an ellipse as diameter, the chords drawn from the extremities of this diameter to the other points where the circle cuts the ellipse, are parallel to the axes.

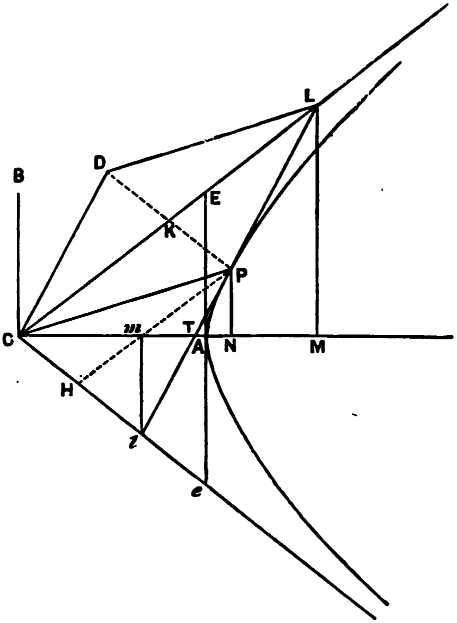
5. Find the least and greatest values of $CP+CD$.

6. No straight line can meet an ellipse or hyperbola at more points than two.

7. Show that a plane can always be found, which passes through the axis minor of an ellipse, and on which the projection of the ellipse is a circle.

8. Show that, with the exception of the case mentioned in Ex. 7, the projection of an ellipse on any plane passing through either axis is an ellipse.

$\therefore$ the complete parallelogram $= 4AC.BC$.



If Ex. 1, page 77, be assumed, the proposition may be demonstrated similarly to that for the ellipse.

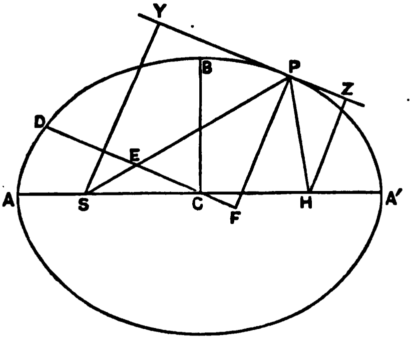
Cor. 1. It should be observed that we have proved that tangents to a hyperbola intercept on the asymptotes, triangles of constant area.

Cor. 2. If PH, PK be drawn respectively parallel to one asymptote and meeting the other, it may be shown that the rectangle $PH.PK = \frac{1}{4}(AC^2 + BC^2)$. For $CEe = CLl$, $\therefore \frac{CL}{CE} = \frac{Ce}{Cl}$ (Euc. VI. 15), $\therefore CL.Cl = CE.Ce = CE^2 = AC^2 + BC^2$; and $CL = 2PH$, and $Cl = 2PK$; $\therefore 4PH.PK = AC^2 + BC^2$.

Cor. 3. As in the ellipse, if PF be perpendicular to CD, it may be proved that $PF.CD = AC.BC$.

PROPOSITION XX.

The semi-diameter, parallel to the tangent at any point on an ellipse, is a mean proportional between the focal distances of that point. $SP.HP=CD^2$.



Let CD be the semi-diameter parallel to the tangent at P, and let SP, and the normal at P, meet CD at E and F respectively. Draw SY, HZ perpendicular to the tangent at P.

Then since angle $HPZ=SPY$ (Prop. 7, Cor. 1), and $CEP=SPY$ (Euc. I. 29), and the angles at Y, Z and F are right angles, therefore the triangles SPY, HPZ, PEF are equiangular,

$$\therefore \frac{SP}{SY}=\frac{HP}{HZ}=\frac{PE}{PF} \quad \text{(Euc. VI. 4)};$$

$$\therefore \frac{SP.HP}{SY.HZ}=\frac{PE^2}{PF^2},$$

$$\text{or } \frac{SP.HP}{BC^2}=\frac{AC^2}{PF^2} \quad \text{(Prop. 10 and Prop. 10, Cor.)}.$$

But $$AC.BC=PF.CD \quad \text{(Prop. 19, Cor.)},$$

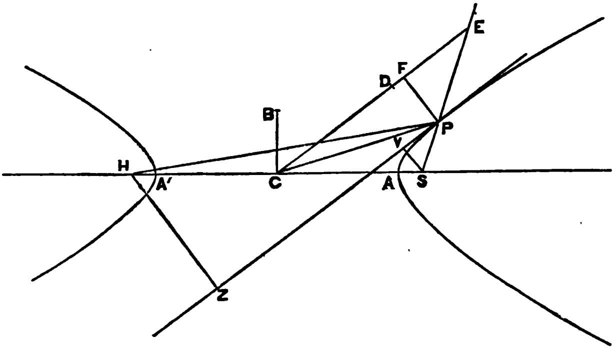
$$\therefore \frac{AC^2}{PF^2}=\frac{CD^2}{BC^2}.$$

Consequently $$\frac{SP.HP}{BC^2}=\frac{CD^2}{BC^2},$$

$$\therefore SP.HP=CD^2.$$

PROPOSITION XX.

The semi-diameter, parallel to the tangent at any point on a hyperbola, is a mean proportional between the focal distances of that point. $SP.HP = CD^2$.



Let CD be the semi-diameter parallel to the tangent at P, and let SP produced, and the normal at P meet CD at E and F respectively. Draw SY, HZ perpendicular to the tangent at P.

Then since angle $HPZ = SPY$ (Prop. 7, Cor. 1), and $CEP = SPY$ (Euc. I. 29), and the angles at Y, Z and F are right angles; therefore the triangles SPY, HPZ, PEF are equiangular,

$$\therefore \frac{SP}{SY} = \frac{HP}{HZ} = \frac{PE}{PF} \quad \text{(Euc. VI. 4)};$$

$$\therefore \frac{SP.HP}{SY.HZ} = \frac{PE^2}{PF^2},$$

$$\text{or } \frac{SP.HP}{BC^2} = \frac{AC^2}{PF^2} \quad \text{(Prop. 10 and Prop. 10, Cor.)}.$$

But $AC.BC = PF.CD$ (Prop. 19, Cor. 3),

$$\therefore \frac{AC^2}{PF^2} = \frac{CD^2}{BC^2}.$$

$$\text{Consequently } \frac{SP.HP}{BC^2} = \frac{CD^2}{BC^2},$$

$$\therefore SP.HP = CD^2.$$

PROBLEMS ON THE ELLIPSE AND HYPERBOLA.

1. Prove in the ellipse or hyperbola that $SX.SC=BC^2$.

2. A person hears the report of a rifle and the impact of the bullet against the target simultaneously, show that he must be at some point of a certain hyperbola. (The path of the bullet is supposed to be straight and its velocity constant.)

3. If PSQ be a focal chord of an ellipse and X the foot of the directrix, show that XP and XQ are equally inclined to the axis.

4. Triangles are described having a given vertical angle at a given point, and having their bases in a given straight line:—the locus of the centres of the circumscribing circles is a hyperbola.

5. A parabola passes through two fixed points and has its axis parallel to a given line:—the locus of the focus of the parabola is a hyperbola.

6. If AB be a straight line whose middle point is C, and P a point such that the rectangle $PA.PB$ together with the square on CP is constant, show that the locus of P is an ellipse.

7. Find the locus of the centre of the circle inscribed in the triangle formed by the transverse axis and the lines joining any point on the hyperbola to its foci.

8. A parabola passes through the foci of an ellipse, and the ellipse through the focus of the parabola; prove that the directrix of the parabola touches the auxiliary circle of the ellipse.

9. P is any point on an ellipse whose foci are S, H: show that if ellipses be described having S and P for foci, and passing through H, the locus of one extremity of the major axis is a circle whose radius is equal to the greatest focal distance in the original ellipse.

10. S, S' are two fixed points; a hyperbola, one of whose foci is S, passes through a given point Q, and touches the line which bisects SS' at right angles, show that its other focus lies on a hyperbola whose foci are S' and Q.

11. For what position of the point P, on an ellipse, is the angle SPH greatest?

12. Show that the axis major of an ellipse is greater than any other diameter.

13. A circle always passes through a given point, and cuts a given straight line at a given angle; prove that the locus of its centre is a hyperbola.

14. If a circle be described touching the axis major of an ellipse at one of the foci, and passing through one extremity of the axis minor, the semi-axis major will be a mean proportional between the diameter of this circle and the semi-axis minor.

15. Show that the points of trisection of a series of conterminous circular arcs lie on branches of two hyperbolas; and determine the distance between their centres.

16. If the centre, a angent, and the length of the transverse axis of an ellipse be given, prove that the directrices pass each through a fixed point.

17. P is a point on a hyperbola whose foci are S and H; another hyperbola is described whose foci are S and P, and whose transverse axis is equal to $SP-2PH$: show that the hyperbolas will meet only at one point, and that they will have the same tangent at that point.

18. A series of ellipses pass through the same point and have a common focus and their major axes of the same length; prove that the locus of their centres is a circle. What are the limits of the eccentricities of the ellipses, and what does the ellipse become at the higher limit?

19. A parabola and hyperbola have the same focus and directrix, and SPQ is a line drawn through the focus S to meet the parabola at P, and the nearer branch of the hyperbola at Q; prove that PQ varies as the rectangle contained by SP and SQ.

20. Find the locus of the centre of a circle which touches externally or internally two given circles. Examine the different cases which arise according to the position and magnitude of the given circles.

21. Find the eccentricity of an ellipse, when the circle that touches it at the extremities of a latus rectum passes (1.) through the centre (2.) through the other focus.

22. If the tangent at a point P of an ellipse meet the axis major at T, and upon CT as diameter a circle be described, and the ordinate NP be produced to meet it at Q, then $CQ=CA$.

23. PG is a normal to an ellipse, terminating at the major axis; the circle of which PG is a diameter cuts SP, HP at K, L respectively: prove that KL is bisected by PG, and is perpendicular to it.

24. If the latus rectum SL of an ellipse meets the auxiliary circle at K, then $SK=BC$.

25. Show that the tangent at the extremity of the latus rectum of an ellipse meets the axis minor produced where the auxiliary circle meets it.

26. The tangent at L, the extremity of the latus rectum of an ellipse meets the axis minor at F, and LH meets it at E, show that E is the centre of the circle described about HLF.

27. If AC be a fixed diameter of a circle, and BD any chord perpendicular to it, the locus of the intersection of AB, DC, is a rectangular hyperbola.

28. What does Ex. 27 become, if the figure is projected?

29. P is any point on an ellipse, AA' its axis major, NP an ordinate to the point P; to any point Q on the curve draw AQ, $A'Q$ meeting NP at R and S; show that $NR.NS=NP^2$.

30. Construct upon the semi-axis minor as base, a rectangle which shall be to the triangle LSH in the duplicate ratio of the major axis to the minor axis, L being the extremity of the latus rectum.

31. If a circle be described passing through any point P of a given hyperbola and the extremities of the transverse axis, and the ordinate NP be produced to meet the circle at Q, show that Q traces out a hyperbola whose conjugate axis is a third proportional to the conjugate and transverse axis of the original hyperbola.

32. The tangent and normal at a point P on an ellipse meet the axis minor at t and g respectively; prove that $Ct.Cg=CS^2$.

33. If the normal at a point P of a hyperbola meet the conjugate axis at g and if S be the focus, then Pg will be to Sg in a constant ratio.

34. The tangent at a point P on an ellipse meets the axes at T and t; show that the angle $PSt=STP$.

35. If the normal to an ellipse at P meet the axis major at G and the axis minor at g, show that the ratio of Gg to Sg is equal to the eccentricity.

36. The tangent at any point P of an ellipse meets the axis minor at T, and the focal distances SP, HP meet it at R, r. Also ST, HT, produced if necessary, meet the normal at P in Q, q respectively. Show that Qr, qR are parallel to the axis major.

37. If one focus, one point, and the length of the major axis of an ellipse be given, show that the ellipse always touches another ellipse having the two given points as foci and whose major axis is less than twice the given axis by the distance between the two given points.

38. P, Q are points on two confocal ellipses at which the line joining the common foci subtends equal angles; prove that the tangents at P, Q are inclined at an angle which is equal to the angle subtended by PQ at either focus.

39. If Y and Z be the feet of the perpendiculars from the foci of an ellipse upon the tangent at any point P, and PN be the ordinate of P, show that the angle $CYZ=HNZ$.

40. The locus of the intersection of the focal distance of any point on an ellipse with the central perpendicular upon the tangent at the same point is a circle.

41. Having given one focus, a tangent, and the length of the minor axis, prove that the locus of the centre is a straight line.

42. If the diameter conjugate to CP meet SP, HP (or these produced) at E, E', prove that $SE=HE'$, and that the circles described about the triangles SCE', HCE are equal.

43. Through N, the foot of the ordinate of a point P on a hyperbola, draw NQ parallel to AP to meet CP at Q, then AQ is parallel to the tangent at P.

44. If two ellipses having equal major axes be inscribed in a parallelogram, the foci of the ellipses will lie in the corners of an equiangular parallelogram.

45. Prove in an ellipse, that if from H a line be drawn parallel to SP, it will meet the perpendicular from S on the tangent at P, in the circumference of a circle.

46. Show that if the distance between the foci of an ellipse be greater than the length of its axis minor, there will be four positions of the tangent, for which the area of the triangle included between it and the straight lines drawn from the centre of the curve to the feet of the perpendiculars from the foci on the tangent, will be the greatest possible.

47. If Y, Z be the feet of the perpendiculars from the foci on the tangent at a point P of an ellipse; prove that the circle circumscribed about the triangle YNZ will pass through C.

48. If S, H be the two foci of an ellipse, and SY the perpendicular from S upon any tangent, prove that HY will bisect the corresponding normal.

49. If from any point P on an ellipse a line be drawn parallel to the axis major, meeting the circle described on the axis minor as diameter at p, and the axis minor at n, prove that

$$\frac{nP}{np}=\frac{AC}{BC}.$$

50. A straight line is drawn through the focus S of an ellipse meeting the two tangents at right angles to it at Y, Y', the diameter parallel to these tangents at L, and the directrix at M: show that $SY.SY'=SL.SM$.

51. Given the foci and the major axis of an ellipse, determine geometrically the point where the tangent is parallel to a given line.

52. In an ellipse with the ordinary notation, show that

$$\frac{PY.PZ}{PN^2}=\frac{SC^2}{BC^2}.$$

53. PT and PN are the tangent and ordinate of a point P on a hyperbola, prove that if T bisects CA then A bisects CN, and if the tangent from P to the auxiliary circle meets it at K, then $PK^2=3AB^2$.

54. The auxiliary circle of an ellipse meets the axis minor produced at K and the ordinate PN at Q, and the tangent to the ellipse at P meets the axis major at T; if KT be joined meeting QPN at R, prove that

$$\frac{RN}{PN}=\frac{QN}{BC}.$$

55. If P be any point on an ellipse whose centre is C and foci S, H; K the point at which the normal at P meets a line at right angles to it through H; E the point of intersection of SP and the diameter conjugate to CP; and EK and CK be joined: then each of the figures $SCKE$, $HCEK$ will be a parallelogram.

56. Given the foci and major axis of an ellipse, obtain by a geometrical construction the points at which the ellipse meets a given straight line drawn through one of the foci.

57. S, H are the foci of an ellipse SY, HZ the perpendiculars on the tangent at P, and a perpendicular from the centre C on the tangent meets SP, HP (produced if necessary) at O, O' respectively; prove that

$$OY=OZ=O'Y=O'Z=SC, \text{ and } SO=HO'=AC.$$

58. Having given the major axis, and a straight line which touches an ellipse, find the point of contact.

59. An ellipse inscribed in a triangle has for one focus the point of intersection of the perpendiculars drawn from the angular points of the triangle on the opposite sides; show that (1.) the other focus is the centre of the circle circumscribing the triangle, and (2.) the major axis of the ellipse is equal to the radius of this circle.

60. From a point T on the auxiliary circle two tangents TP, TQ are drawn to the ellipse, show that the angle POQ is double the angle STH.

61. If the normal at a point P of an ellipse meet the major axis at G and SY, HZ be the perpendiculars on the tangent at P from the foci, and YG be produced to meet ZH produced at Z', show that $HZ'=HZ$.

62. One vertex of each of two similar ellipses coincides with the centre of the other: prove that the lines joining the points of intersection to the remaining vertices are tangents.

63. If S be the focus of an ellipse, X the foot of the corresponding directrix, XYY' a straight line meeting the auxiliary circle at Y, Y', prove that SY, SY' are equally inclined to the axis.

64. Y and Z are the feet of the perpendiculars from the foci on a tangent to an ellipse; if a circle be described through Y and Z and touching the major axis at Q, prove that if the diameter of the circle which passes through Q meets the tangent at K then $KQ=BC$.

65. If AQ be drawn from one of the vertices of an ellipse perpendicular to the tangent at any point P, prove that the locus of the point of intersection of PS and AQ produced will be a circle.

66. PS is the focal distance of a point P on an ellipse; CR is the radius of the auxiliary circle parallel to PS, and drawn in the direction from P to S; SQ is a perpendicular on CR: show that the rectangle contained by SP and QR is equal to the square on half the minor axis.

67. If P, Q be any two points on a fixed ellipse whose foci are S, H, and if SP, HQ intersect within the ellipse at R, prove that two ellipses can be drawn touching each other at R, the one having S for focus and touching the given ellipse at Q, the other having H for focus and touching the given ellipse at P. If the major axis of one of the variable ellipses be given, find the loci of their other foci, and of the point of contact.

68. If NP be the ordinate of a point P on an ellipse, Y and Z the points where the tangent at P meets the perpendiculars from the foci, then

$$\frac{NY}{NZ}=\frac{PY}{PZ}.$$

69. If from the centre of an ellipse lines be drawn parallel and perpendicular to the tangent at any point, they enclose a part of one of the focal distances of that point which shall be equal to the other focal distance.

70. If any ordinate NP of an ellipse be produced to meet the tangent at the extremity of the latus rectum at Q, prove that $QN=PS$.

71. Prove that an ellipse can be inscribed in any parallelogram, so as to touch the four sides at their middle points.

72. Two conjugate diameters of an ellipse are cut by the tangent at any point P at M, N; prove that the area of the triangle CPM varies inversely as that of CPN.

73. Employ the method of projections to show that the locus of the middle points of all chords in an ellipse which pass through a fixed point is an ellipse similar and similarly situated to the given ellipse and with its centre in the middle point of the line joining the given point and the centre of the given ellipse.

74. Find the greatest isosceles triangle which can be inscribed in an ellipse with its base parallel to the major axis.

75. Determine the greatest triangle which can be inscribed in an ellipse having an angular point fixed at a point in its perimeter.

76. If two tangents to an ellipse and the chord of contact include a constant area, prove that, (1.) the chord of contact always touches a concentric similar ellipse, (2.) that the intersection of the tangents lies on another concentric similar ellipse.

77. If the diagonals of a quadrilateral inscribed in an ellipse meet at the centre, the quadrilateral is a parallelogram.

78. If the diagonals of a quadrilateral circumscribing an ellipse meet at the centre, the quadrilateral is a parallelogram.

79. Prove that the parallelograms whose sides touch an ellipse at the extremities of conjugate diameters have the least area of all which circumscribe the ellipse.

80. If the sides of the triangle ABC touch an ellipse at the points P, Q, R, then AP, BQ, CR meet at a point.

81. If through the extremity A of the major axis of an ellipse a line be drawn parallel to a diameter CP, meeting the conjugate diameter CD at V; and if PN be the ordinate of P, prove that the triangle $ACV = CPN$.

82. TQ is a tangent to an ellipse at Q and $TPCP'$ a secant, meeting the ellipse at P and P' and passing through the centre C; the tangent at P meets TQ at L, and $P'Q$ and PL produced meet at R: show that PR is bisected at L. Hence if PP' become the axis major, find a simple geometrical construction for drawing a tangent to an ellipse at any given point Q.

83. The circle, described on the focal distance of any point on an ellipse as diameter, touches the auxiliary circle.

84. From the point of intersection of an asymptote and a directrix of a hyperbola, a tangent is drawn to the curve, prove that the line joining the point of contact with the focus is parallel to the asymptote. Also show that this focal distance is equal to one fourth of the latus rectum.

85. If a hyperbola and its conjugate are described having a pair of conjugate diameters of an ellipse for asymptotes, and cutting the ellipse at the points a, b, c, d taken in order; show that the diameters Ca, Cc, and Cb, Cd are conjugate diameters in the ellipse; also that Ca, Cd, and Cb, Cc are conjugate diameters in the hyperbola: C being the common centre of the curves.

86. Given one asymptote of a hyperbola, the direction of the other and the position of one focus, determine the position of the vertices.

87. If the tangent at a point P of a hyperbola meet one asymptote at T, and a line TQ be drawn parallel to the other asymptote to meet the curve at Q, prove that if PQ be joined and produced both ways to meet the asymptotes at R and R', RR' will be trisected at P, Q.

88. Prove that the angles subtended at the vertices of a rectangular hyperbola by any chord parallel to the conjugate axis, are supplementary.

89. The tangent at a point P of a rectangular hyperbola meets a diameter QCQ' at T: show that CQ and TQ' subtend equal angles at P.

90. In the rectangular hyperbola, prove that the triangle formed by the tangent at any point and its intercepts on the axes, is similar to the triangle formed by the straight line joining that point with the centre, and the abscissa and ordinate of the point.

91. Two tangents are drawn to the same branch of a rectangular hyperbola from an external point; prove that the angles which these tangents subtend at the centre are respectively equal to the angles which they make with the chord of contact.

92. If in an equilateral hyperbola QV be an ordinate to a diameter CP, and the tangent at Q meet CP at T, show that the circle circumscribing CTQ touches QV.

93. Show that the normals at the extremities of two conjugate diameters of a rectangular hyperbola intersect on the asymptote, and are parallel to another pair of conjugate diameters.

94. A parallelogram is inscribed in a given ellipse; show that its sides are parallel to conjugate diameters.

95. A parallelogram is described about an ellipse, prove that its diagonals are conjugate to each other.

96. An ellipse being drawn, find its axes.

97. If CQ be conjugate to the normal at P in an ellipse, then is CP conjugate to the normal at Q.

98. If PM, PN be drawn parallel to the asymptotes CN, CM of a hyperbola, and an ellipse be constructed having CN, CM for semi-diameters; prove that if CP cuts the ellipse at Q, the tangents at Q and P to the ellipse and hyperbola are parallel.

99. PCP' is any diameter of an ellipse; the tangents at any two points D and E intersect at F, and PE, $P'D$ intersect at G: show that FG is parallel to the diameter conjugate to PCP'.

100. CP, CD are conjugate semi-diameters of an ellipse, such that CP is perpendicular to AB, and PN, DR are ordinates, prove that $PN.DR=CN^2$.

101. The normals at P and D, the extremities of conjugate diameters of an ellipse, intersect the major and minor axes at G, K, G', K': prove that the triangle $CGK=CG'K'$.

102. From any point P on an ellipse two chords are drawn at right angles, the line joining the other extremities of the chords being parallel to the tangent at P, then the chords bisect the angles between the tangent and diameter at P.

103. Prove that if two hyperbolas have their transverse axes parallel, and their eccentricities equal, they will have parallel asymptotes. Does the converse proposition hold?

104. Any pair of conjugate diameters divides the ellipse into four equal elliptic sectors.

105. If the elliptic area between the two radii CP, CQ is constant, prove that the area of the segment PQ is constant.

106. If P be a point on a hyperbola, and PM, PN be drawn parallel to the asymptotes CN, CM, and if an ellipse be constructed having CN, CM for semi-diameters, prove that if CP cuts the ellipse at Q, the tangents at Q and P to the ellipse and hyperbola are parallel.

107. If SVs, TVt be two tangents to a hyperbola cutting one asymptote at S, T, and the other at s, t, prove that

$$\frac{VS}{Vs}=\frac{Vt}{VT}.$$

108. CP and CD are conjugate semi-diameters of an ellipse; PQ is a chord parallel to one of the axes: show that DQ is parallel to one of the straight lines which join the ends of the axes.

109. If CP, CD be conjugate semi-diameters of an ellipse, prove that PD is proportional to the diameter parallel to it.

110. Show that if CN, NP are the abscissa and ordinate of any point P in a circle, and NQ is taken equal to NP, and inclined to it at a constant angle, the locus of the point Q is an ellipse; and determine the axes.

111. Find the position and magnitude of the axes of a hyperbola which has a given line for asymptote, passes through a given point and touches a given straight line at a given point.

112. If the focal distance SP of any point P meet the diameter conjugate to CP at E, then the difference of the squares on CP and SE will be equal to the square on BC.

113. Supposing the two asymptotes and one point of a hyperbola to be given in position, show how to construct the curve; and find the position of the foci.

114. If in an ellipse CD be conjugate to CP, and PG be the normal at P; and if DR, PN be ordinates, prove that the triangles PGN, DRC are similar; and hence deduce that PG bears a constant ratio to CD.

115. Through any point D on an ellipse, a straight line is drawn perpendicular to the radius vector CD meeting the auxiliary circle at R, R': prove that RR' is equal to the difference of the focal distances of the extremity of the conjugate diameter CP.

116. If from a point on a hyperbola PH, PK be drawn parallel to and terminated by the asymptotes, and a line through the centre meet PH, PK at R, T, and the parallelogram $PRQT$ be completed, Q is a point on the hyperbola.

117. A series of ellipses are confocal with a given hyperbola: prove that either focal distance of any point of intersection is cut by its conjugate diameter with respect to the particular ellipse in a point which lies on a circle.

118. OQ, OQ' are tangents to an ellipse and the parallelogram $OQRQ'$ is completed; prove that if R be on the ellipse, O will lie on a similar and similarly situated concentric ellipse.

119. The tangent at any point on a hyperbola is produced to meet the asymptotes, thus forming a triangle: determine the locus of the point of intersection of the straight lines drawn from the angles of this triangle to bisect the opposite sides.

120. The tangent and normal at any point of a hyperbola intersects the asymptotes and axis respectively at four points which lie on a circle passing through the centre of the hyperbola. Show also that the radius of this circle varies inversely as the perpendicular from the centre upon the tangent.

121. In a hyperbola, when is $CP \sim CD$ greatest?

122. Determine the locus of the middle points of chords of an ellipse which bear a constant ratio to the diameter parallel to them.

123. Tangents are drawn to a hyperbola, and the portion of each tangent intercepted by the asymptotes is divided in a given ratio; prove that the locus of the point of section is a hyperbola.

124. Ellipses are described on AB as diameter and touching BC; if tangents be drawn to them from C, show that the locus of the points of contact is a straight line.

125. PQ, PQ' are tangents to an ellipse whose centre is C; if each point of contact be joined with the extremity of the diameter drawn through the other point, then these two chords will be each parallel to CP.

126. If the tangent at a point P of an ellipse meets the axis minor at t, and St be joined, and BE be drawn through B parallel to St to meet the axis major at E, show that BE is equal to the semi-diameter conjugate to CP.

127. If P is one of the points on an ellipse where SP, HP are at right angles, and CD the diameter conjugate to CP meets SP, HP (produced if necessary) at E, E', show that

$$\frac{EE'}{CD}=\frac{AC}{BC}.$$

128. If P be a point on a hyperbola such that SP, HP are at right angles, show that the square on the diameter conjugate to CP is double that on the conjugate axis.

129. If CP, CD be conjugate semi-diameters in an ellipse, and the normal at P meet the axis major at G, then

$$SG.HG=CD^2-PG^2.$$

130. Q is a point in one asymptote of a hyperbola and q in the other; if Qq move parallel to itself, find the locus of the point of intersection of tangents to the hyperbola from Q and q.

131. In the figure to Prop. 19 of the hyperbola, show that PH passes through m, and that KP produced passes through M.

CHAPTER V.

CONICS.

It will have been noticed that many of the Properties which have been investigated in the previous chapters are common to the three curves. The properties which are thus true for all conics are those which refer to one focus and directrix only. The ellipse and hyperbola are separated from the parabola by their possession of a centre and two foci and directrices. They are consequently called *Central Conics.* The parabola may, however, be regarded as a particular case of the ellipse or hyperbola whose centre and further focus have moved off to an infinite distance, just as the circle is a particular case of the ellipse whose foci coincide with the centre, and whose directrices are at an infinite distance.

We purpose in the present chapter to give the principal propositions which are true for all conics.

PROPOSITION I.

If the straight line joining two points P, P' on a conic meet the directrix at K, the straight line joining K with the focus S will bisect the exterior angle between PS and $P'S$.

This has been proved for the parabola in Prop. 3, and for the ellipse and hyperbola in Prop. 5. The proof given for the ellipse will hold for any conic. It should be observed that the Proposition is not strictly true for the hyperbola unless the two points are on the same branch of the curve. (See Hyperbola, Prop. 5.)

PROPOSITION II.

The portion of a tangent to a conic intercepted between the point of contact and the directrix subtends a right angle at the focus.

This has been proved for the parabola in Prop. 3, and for the ellipse and hyperbola in Prop. 5, Cor. The proof for the ellipse will hold for any conic.

PROPOSITION III.

The tangents at the extremities of any focal chord of a conic intersect on the directrix.

This has been proved for the parabola in Prop. 4, and a similar proof will hold for any conic. It must, however, be observed that the two tangents are not at right angles except in the parabola.

PROPOSITION IV.

If the normal at a point P on a conic meet the axis at G, then the ratio of SG to SP is equal to the eccentricity of the conic.

This has been proved for the parabola in Prop. 3, Cor. 4, and for the ellipse and hyperbola in Prop. 7. Either of the latter proofs will hold for any conic.

PROPOSITION V.

The middle points of any series of parallel chords of a conic lie on a straight line.

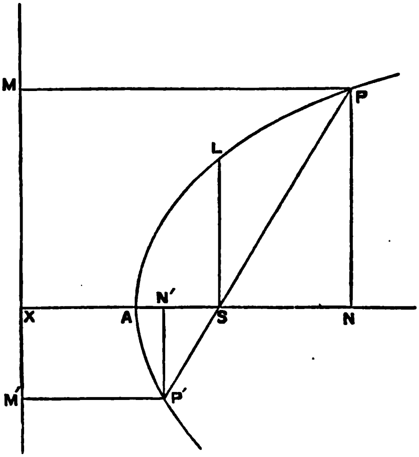
This has been proved for the parabola in Prop. 8, and for the ellipse and hyperbola in Prop. 16.

It has also been proved in these propositions that the straight line which is the locus of the middle points of a series of parallel chords is a diameter of the conic, and that the tangent at the point where it meets the conic is parallel to the series of chords.

PROPOSITION VI.

The semi-latus rectum of a conic is a harmonic mean between the segments of any focal chord. $\frac{1}{SP}+\frac{1}{SP'}=\frac{2}{SL}$.

This has been proved for the parabola in Prop. 15. The following generalizes the proof for any conic.



Let PSP' be a focal chord and SL the semi-latus rectum Draw the ordinates PN, $P'N'$, and also draw PM, $P'M'$ perpendicular to the directrix. Then from the similar triangles $SP'N'$ SPN we have $\frac{SN'}{SP'}=\frac{SN}{SP}$, or $\frac{SX-N'X}{SP'}=\frac{NX-SX}{SP}$;

$$\therefore \frac{SX}{SP'}+\frac{SX}{SP}=\frac{NX}{SP}+\frac{N'X}{SP'}=\frac{PM}{SP}+\frac{P'M'}{SP'}$$

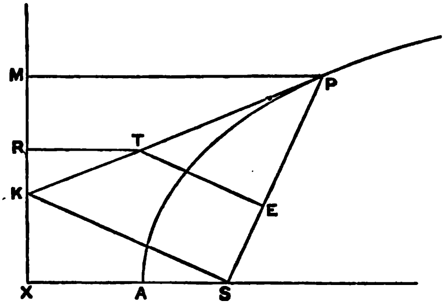
$$=\frac{SX}{SL}+\frac{SX}{SL} \quad \text{(Def. of Conic).}$$

$$\therefore \frac{1}{SP'}+\frac{1}{SP}=\frac{2}{SL}.$$

PROPOSITION VII.

If from any point T on the tangent at a point P of a conic, perpendiculars TE, TR be drawn to SP and the directrix respectively, then the ratio of SE to TR is equal to the eccentricity of the conic.

Let the tangent at P meet the directrix at K. Join SK and draw PM perpendicular to the directrix.



Then since KS is at right angles to SP (Prop. 2), $\therefore$ KS is parallel to TE (Euc. I. 28). Also the triangles KTR and KPM are equiangular,

$$\therefore \frac{TR}{PM}=\frac{TK}{PK} \quad \text{(Euc. VI. 4)}$$

$$=\frac{SE}{SP} \quad \text{(Euc. VI. 2)};$$

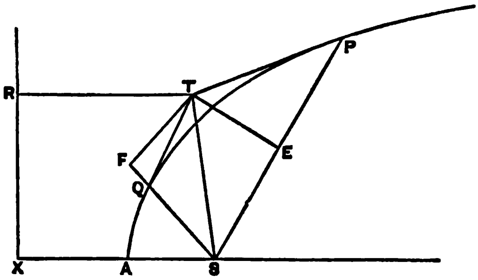
$$\therefore \frac{SE}{TR}=\frac{SP}{PM}= \text{the eccentricity of the conic.}$$

This Proposition is due to Professor J. C. Adams, and is known by his name.

Cor. Similarly it may be shown that if T be any point on a secant PP' of a conic, and if TEE' be drawn making equal angles with SP and SP', and TR be perpendicular to the directrix, then the ratio of SE or SE' to TR is equal to the eccentricity.

Proposition VIII.

Tangents drawn from an external point to a conic subtend equal angles at the focus.



Let TP, TQ be two tangents drawn from a point T to the conic. Join SP, SQ, ST, and draw TE, TF, TR perpendicular to SP, SQ, and the directrix respectively. Then each of the ratios $\frac{SE}{TR}$, $\frac{SF}{TR}$ is equal to the eccentricity (Prop. 7), and $\therefore$ $SE=SF$.

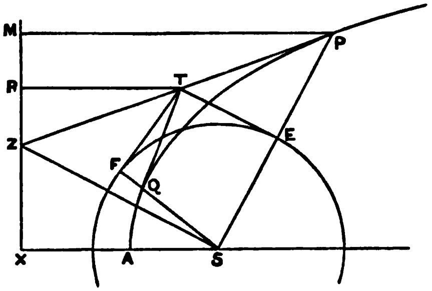
Again, since $SE=SF$ and ST is common to the two triangles SET, SFT, and the angles at E and F are right angles, $\therefore$ angle $TSE=TSF$, that is TP, TQ subtend equal angles at S.

If the figure be drawn in the case of the hyperbola where the tangents touch different branches of the curve, it may be shown that the angles they subtend at the focus are supplementary.

PROPOSITION IX.

To draw a pair of tangents to a conic from a given external point.

Let T be the given point. Draw TR perpendicular to the directrix. With centre S describe a circle whose radius is a fourth proportional to AX, AS and TR. Draw the tangents TE, TF to the circle and join SE, SF, producing them if necessary to meet the conic at P, Q; then TP, TQ shall be tangents to the conic. Let PT meet the directrix at Z; join SZ and draw PM perpendicular to the directrix. Then by construction



$$\frac{TR}{SE}=\frac{AX}{AS}=\frac{PM}{SP} \quad \text{(Def. of Conic)},$$

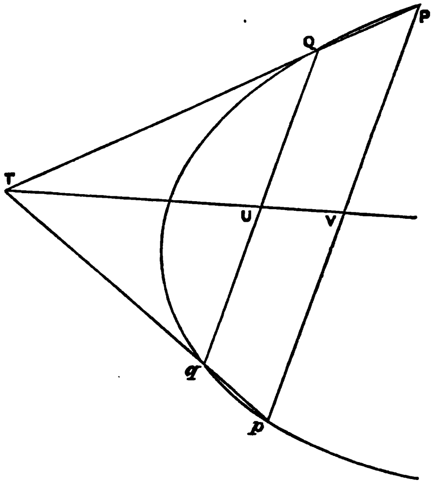
$$\therefore \frac{SP}{SE}=\frac{PM}{TR}=\frac{PZ}{TZ} \quad \text{(Euc. VI. 4)}.$$

$\therefore$ SZ is parallel to TE (Euc. VI. 2). But TE is at right angles to SE (Euc. III. 18). $\therefore$ ZSP is a right angle and consequently ZTP is a tangent to the conic (Prop. 2). Similarly it may be shown that TQ is a tangent.

NOTE. The student ought to observe how this construction would apply in the case of the hyperbola when the point T lies between the asymptotes and the conjugate axis.

PROPOSITION X.

The tangents at the extremities of any chord of a conic intersect on the diameter which bisects the chord.



Let PVp, QUq be two parallel chords of a conic, and let TUV be the diameter which bisects them. Join PQ and produce it to meet this diameter at T. Then from similar triangles we have

$$\frac{TU}{TV}=\frac{QU}{PV}$$

$$=\frac{qU}{pV},$$

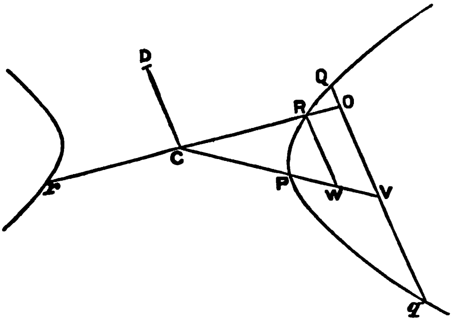
∴ Tqp is a straight line (Euc. VI. 32).

We have thus proved that the lines joining the extremities of a pair of parallel chords of a conic intersect on the diameter which bisects the chords; and this will be true in the limit when Qq moves up and ultimately coincides with Pp; in which case TP and Tp become the tangents at P, p, and they intersect on the diameter bisecting Pp.

Proposition XI.

If through any point two chords be drawn to a conic, parallel to two fixed lines, the ratio of the rectangles of the segments of these chords is constant.

This has been proved for the parabola in Prop. 22. The property may easily be proved in the case of the ellipse by using the method of projections; or the following, which is the proof for the hyperbola, will with the proper figure hold for either curve.



Let QOq be one of the chords drawn through the point O parallel to the fixed diameter CD, and let CPV be the diameter bisecting Qq at V, so that CP and CD are conjugate. Draw the diameter $rCRO$ through O, and draw RW parallel to QV, meeting CV at W.

Then $$\frac{QV^2}{CD^2}=\frac{CV^2-CP^2}{CP^2} \quad \text{(Prop. 17, Chap. IV.)},$$

or $$\frac{QV^2+CD^2}{CD^2}=\frac{CV^2}{CP^2}.$$

Similarly $\dfrac{RW^2+CD^2}{CD^2}=\dfrac{CW^2}{CP^2}$,

$$\therefore \frac{QV^2+CD^2}{RW^2+CD^2}=\frac{CV^2}{CW^2}=\frac{CO^2}{CR^2} \quad \text{(Euc. VI. 2)};$$

$$\therefore \frac{QV^2+CD^2}{CO^2}=\frac{RW^2+CD^2}{CR^2}.$$

But
$$\frac{RW^2}{CR^2}=\frac{OV^2}{CO^2} \quad \text{(Euc. VI. 4)},$$

$$\therefore \frac{QV^2+CD^2}{CO^2}=\frac{OV^2}{CO^2}+\frac{CD^2}{CR^2};$$

$$\frac{QV^2+CD^2-OV^2}{CO^2}=\frac{CD^2}{CR^2},$$

$$\therefore \frac{QV^2-OV^2}{CO^2-CR}=\frac{CD^2}{CR^2},$$

or
$$\frac{OQ.Oq}{OR.Or}=\frac{CD^2}{CR^2} \quad \text{(Euc. II. 5, 6)}.$$

Similarly it may be shown that if $Q'Oq'$ be the chord through O parallel to a fixed diameter CD',

$$\frac{OQ'.Oq'}{OR.Or}=\frac{CD'^2}{CR^2};$$

$$\therefore \frac{OQ.Oq}{OQ'.Oq'}=\frac{CD^2}{CD'^2};$$

which shows that the ratio of these rectangles is independent of the position of the point O.

COR. 1. The proposition will be true in the limit when the chords become the two tangents TP, TP' parallel to the fixed diameters CD, CD' respectively, in which case we have

$$\frac{OQ.Oq}{OQ'.Oq'}=\frac{TP^2}{TP'^2}=\frac{CD^2}{CD'^2}.$$

COR. 2. It may be shown for any conic in a manner similar to Prop. 23 for the parabola, that if a circle be drawn cutting a conic at four points the three pairs of common chords are respectively equally inclined to the axis of the conic.

PROBLEMS ON CONICS.

1. Having given two points on a conic, the focus and the eccentricity, describe the conic.

2. Having given two points on a conic, the directrix and the eccentricity, describe the conic.

3. Show that a straight line cannot cut a conic at more points than two.

4. The angle which the tangent to a conic makes with the focal distance of the point of contact is greater than, equal to, or less than the angle it makes with the perpendicular on the directrix, according as the conic is an ellipse, parabola, or hyperbola.

5. If from G the foot of the normal at any point P on a conic, GK be drawn perpendicular to SP, PK is equal to the semi-latus rectum.

6. Two conics have a common focus and touch each other; from any point on their common tangent, the other tangents are drawn to both: prove that the line joining their points of contact passes through the focus.

7. The normal at a point P of an ellipse meets the ellipse again at Q, and the tangents at P and Q intersect at T; TS meets PQ at R, and HP, SQ meet at O: prove that OR bisects the angle POS.

8. A parabola is described with its focus at one focus of a given conic and touches the conic: prove that its directrix will touch a fixed circle.

9. From any point T in the tangent at P to a conic, two straight lines are drawn perpendicular to SP and the major axis, the former meeting SP at Q and the latter meeting the conic at R; prove that $SQ=SR$.

10. A circle is described about the focus of a hyperbola as centre, with a radius equal to one-fourth of the latus-rectum: prove that the focal distances of the points at which it intersects the hyperbola are parallel to the asymptotes.

11. Two points on a conic and the angle subtended at the focus by the line joining them are given; show that the line joining the focus with the intersection of the tangents at the points always passes through a fixed point.

12. A tangent to a conic at a point P intersects a fixed tangent at T; if S is the focus and a line be drawn through S perpendicular to ST, meeting the tangent at P at Q, show that the locus of Q is a straight line touching the conic.

13. From any point T outside an ellipse two tangents TP and TQ are drawn to the ellipse; show that a circle can be described with T as centre so as to touch SP, HP, SQ, HQ, or these lines produced if necessary.

14. SY is perpendicular from the focus on the tangent at any point P on a conic, and T is the point where the tangent meets the directrix: show that the perpendicular from Y on ST is a third proportional to SP and SY.

15. PSQ is any focal chord of a conic; the normals at P and Q intersect at K, and KN is drawn perpendicular to PQ, prove that $PN = SQ$. Hence deduce the locus of N.

16. Given a focus of a conic inscribed in a given triangle, find the points where it touches the sides.

17. If P, Q be two points on a conic and p, q two points on the directrix, such that pq subtends at the focus half the angle subtended by PQ, either Pp and Qq or Pq and Qp meet on the curve.

18. Two variable tangents to a conic intercept on a fixed tangent a length which subtends a constant angle at the focus: show that the locus of their intersection is a conic with the same focus and directrix.

19. Tangents are drawn at two fixed points P, P' on a conic: if any tangent be drawn meeting those at P, P' at R, R', show that the line bisecting the angle RSR' intersects RR' on a fixed tangent to the conic: find the point of contact of this tangent.

20. Prove that focal chords of a central conic are in the ratio of the squares on the parallel diameters.

21. Tangents are drawn to a central conic from a point O, and the line joining the points of contact meets the directrices at R and R'; if SR, HR' meet at O', prove that the axis minor bisects OO'.

22. If P, Q be the extremities of a focal chord of a conic, PM, QN perpendiculars on the directrix, then PN, QM meet on the axis; and, if the vertex be the meeting point, the curve will be a parabola.

23. If AB, CD, two chords of a conic, not parallel to one another make equal angles with either axis; the lines AC, BD and AD, BC will also make equal angles with either axis.

24. The normals at the extremities of a focal chord PSQ of a conic intersect at K, and KL is drawn perpendicular to PQ; KF is a diagonal of the parallelogram of which SK, KL are adjacent sides: prove that KF is parallel to the transverse axis of the conic.

CHAPTER VI.

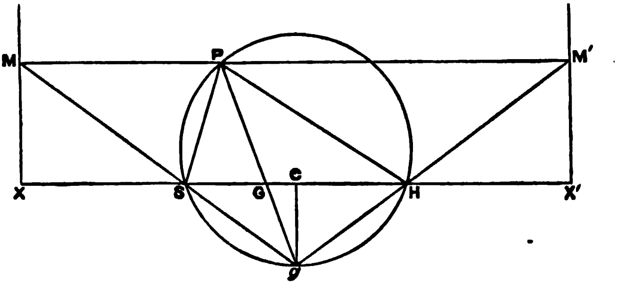
CENTRAL CONICS.

We propose in the present Chapter to demonstrate some further properties of the ellipse and hyperbola in addition to those already proved in chapters III. and IV.

The properties which were established for the ellipse and hyperbola in Proposition 3, namely, that in the former curve, the sum, and in the latter, the difference of the focal distances of any point is constant, are sometimes given as the definitions of the two curves. If these are adopted, the two following Propositions are necessary in order to establish the definitions which we have assumed.

Proposition I.

If a plane curve be traced out by a point which moves in such a manner that the sum of its distances from two fixed points is constant, then the distance of any point on the curve from one of these fixed points bears a constant ratio, less than unity, to its perpendicular distance from a fixed straight line.



Let S, H, be the two fixed points, and P any point on the curve. Join SP, HP, and describe a circle about the triangle SPH. Bisect SH at right angles by Cg which meets the circle on the side of SH remote from P, at g. Join Pg meeting SH at G, and also Sg, Hg; and through P draw a line parallel to SH meeting gS, gH produced at M, M'. Draw MX, $M'X'$ perpendicular to SH produced.

Then since Cg bisects SH at right angles, $gS=gH$ and angle $gSH=gHS$ (Euc. I. 4); and since MM' is parallel to SH $\therefore$ $gM=gM'$, and hence it follows that $CX=CX'$.

Now the angle $MPS=PSH$ (Euc. I. 29)

$=PgH$ (Euc. III. 21).

Also angle $PMS=gSH=gHS$ (Euc. I. 29)

$=gPH=gPS$ (Euc. III. 21).

Therefore the triangles SPM, GSP, and HgP are equiangular.

In the same manner it may be shown that the triangles HPM', GHP, and SgP are equiangular:

$$\therefore \frac{SP}{PM}=\frac{SG}{SP}=\frac{Hg}{Pg}=\frac{Sg}{Pg} \text{ (Euc. VI. 4).}$$

$$\text{Also } \frac{HP}{PM'}=\frac{HG}{HP}=\frac{Sg}{Pg} \text{ (Euc. VI. 4).}$$

$$\therefore \frac{SP}{PM}=\frac{HP}{PM'}=\frac{SG}{SP}=\frac{HG}{HP}$$

$$=\frac{SP+HP}{MM'}=\frac{SH}{SP+HP}.$$

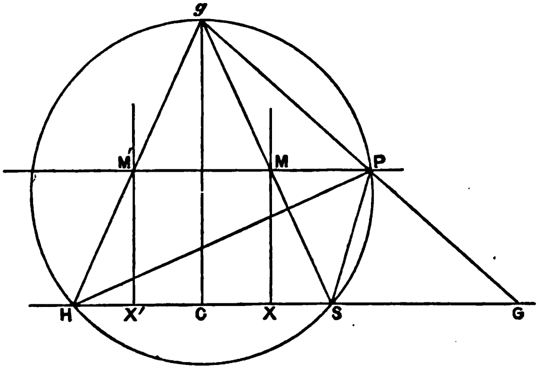
Therefore since $SP+HP$ is constant, MM' or $2CX$ is also constant, $\therefore$ MX is a fixed straight line. Also the ratio

$$\frac{SP}{PM}=\frac{SH}{SP+HP}, \text{ which is constant and less than unity.}$$

PROPOSITION II.

If a plane curve be traced out by a point which moves in such a manner that the difference of its distances from two fixed points is constant, then the distance of any point on the curve from one *of these fixed points bears a constant ratio greater than unity, to its perpendicular distance from a fixed straight line.*

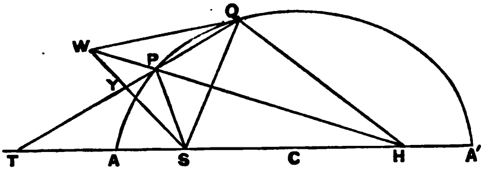
With the figure below, the proof will be found to be almost identical with that in the previous proposition.



Proposition III.

The straight line which bisects the exterior angle between the focal distances of any point on an ellipse does not meet the curve again.

Let *PT* be the straight line which bisects the angle between *SP* and *HP* produced, and if possible let it meet the curve again at *Q*. Draw *SY* perpendicular to *PT*, and produce it to meet *HP* produced at *W*. Join *QW*, *QS*, *QH*.



Then since the angles at *Y* are right angles, and angle

$SPY=WPY$ and PY is common to the two triangles SPY, WPY $\therefore$ $SP=WP$ and $SY=WY$ (Euc. I. 26).

$\therefore$ $SP+HP=WP+HP=HW=2AC$ (Prop. 3, chap. III.). Again, since $SY=WY$ and QY is common to the two triangles SYQ and WYQ, and at right angles to SW,

$\therefore$ $SQ=WQ$ (Euc. I. 4);
$\therefore$ $SQ+HQ=WQ+HQ=2AC$ (Prop. 3. chap. III.),
$\therefore$ $WQ+HQ=WH$, which is impossible.

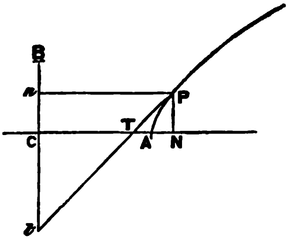
Hence the line bisecting SPW cannot meet the curve again.

A similar proof will hold for the hyperbola, remembering that the word "exterior" must be omitted from the above enunciation.

PROPOSITION IV.

If the tangent at a point P of an ellipse or hyperbola meet the conjugate axis BC at t, and Pn be drawn perpendicular to BC then $Cn.Ct=BC^2$.

This may be proved very simply in the case of the ellipse by projections, or by the following method, which is the proof for the hyperbola.



Draw the ordinate PN, and let Pt meet AC at T, then from similar triangles,

$$\frac{Ct}{CT}=\frac{PN}{TN}, \quad \therefore \quad \frac{Cn.Ct}{CN.CT}=\frac{Cn.PN}{CN.TN}$$
$$=\frac{PN^2}{CN^2-CN.CT} \quad \text{(Euc. II. 3)};$$

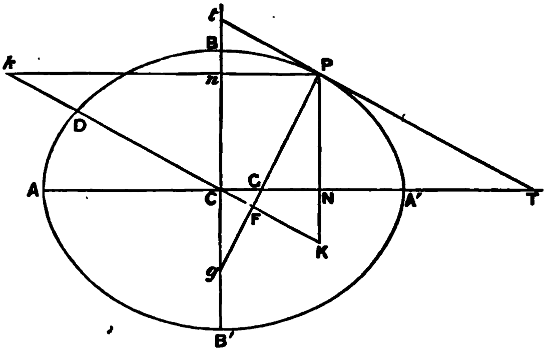
$$\text{or}\quad \frac{Cn.Ct}{AC^2}=\frac{PN^2}{CN^2-AC^2}\quad \text{(Prop. 8, chap. IV.)}$$

$$=\frac{BC^2}{AC^2}\quad \text{(Prop. 6, chap. IV.)};$$

$$\therefore\quad Cn.Ct=BC^2.$$

Proposition V.

If the normal at a point P of an ellipse or hyperbola meet the transverse axis at G, the conjugate axis at g, and the diameter CD parallel to the tangent at P, at F, then $PG.PF=BC^2$ and $Pg.PF=AC^2$.



Considering either figure, let the tangent at P meet the axes at T, t. Draw the ordinates PN, Pn and produce them to meet CD at K, k. Then since the angles at N and F are right angles, the triangles PNG, PFK are equiangular,

$$\therefore\quad \frac{PG}{PN}=\frac{PK}{PF}\quad \text{(Euc. VI. 4)},$$

$$\therefore\ PG.PF=PN.PK$$

$$=Cn.Ct\ \text{(Euc. I. 34)}$$

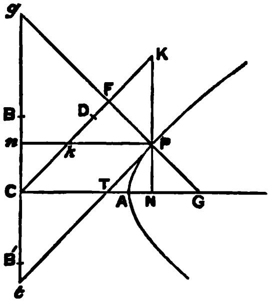
$$=BC^2\ \text{(Prop. 4)}.$$

Similarly from the triangles Png, PFk we get

$$Pg.PF=Pn.Pk$$

$$=CN.CT\ \text{(Euc. I. 34)}$$

$$=AC^2\ \text{(Prop. 8, chap. III. and IV.)}.$$



Cor. Hence it may be proved that $PG.Pg = CD^2$.

For $PG.PF = BC^2$,

$$\therefore \quad \frac{PG}{BC} = \frac{BC}{PF} = \frac{BC.CD}{PF.CD}$$

$$= \frac{BC.CD}{AC.BC} = \frac{CD}{AC} \quad \text{(Prop. 19, Cor. chap. III., IV.).}$$

Similarly it may be shown that

$$\frac{Pg}{AC} = \frac{CD}{BC};$$

$$\therefore \quad \frac{PG.Pg}{AC.BC} = \frac{CD^2}{AC.BC},$$

$$\text{or} \quad PG.Pg = CD^2.$$

Proposition VI.

If the normal at a point P of an ellipse or hyperbola meet the transverse axis at G, and N be the foot of the ordinate, then $\frac{GN}{CN} = \frac{BC^2}{AC^2}$ *and* $\frac{CG}{CN} = \frac{CS^2}{AC^2}$.

With the figures of the previous Prop., since GPT is a right angle, the triangle GPN is equiangular to PTN and therefore to tTC. Hence

$$\frac{GN}{PN} = \frac{Ct}{CT} \quad \text{(Euc. VI. 4),}$$

$$\therefore \quad \frac{GN.Cn}{PN.CN} = \frac{Ct.Cn}{CT.CN},$$

$$\therefore \quad \frac{GN}{CN} = \frac{BC^2}{AC^2} \text{ (Prop. 8, chap. III., IV. and Prop. 4);}$$

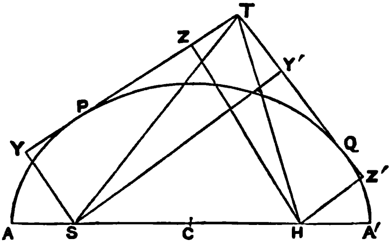
$\therefore \dfrac{CN \mp GN}{CN} = \dfrac{AC^2 \mp BC^2}{AC^2}$, the upper or lower signs being taken for the ellipse or hyperbola respectively;

$$\therefore \quad \frac{CG}{CN} = \frac{CS^2}{AC^2} \text{ (Prop. 4, chap. III., IV.).}$$

PROPOSITION VII.

Tangents TP, TQ drawn from an external point T to an ellipse or hyperbola make equal angles with ST, HT respectively.

Taking the ellipse, draw SY, HZ perpendicular to TP; and SY', HZ' to TQ.



Then $SY.HZ = SY'.HZ' = BC^2$ (Prop. 10, chap. III., IV.),

$$\therefore \quad \frac{SY}{SY'} = \frac{HZ'}{HZ}.$$

Hence the angle PTQ is divided into two parts by ST, and also into two parts by HT, so that the ratio of the perpendiculars from S on TP, TQ is equal to the ratio of the perpendiculars from H on TQ, TP; and since a given angle can be divided into two parts, such that the ratio of the perpendiculars from any point in the dividing line on the two legs of the angle may be a fixed ratio, in one way only, it follows that the angle $STP = HTQ$.

The proof for the hyperbola is the same, it being noticed that when the two tangents touch the same branch of the curve, ST makes the same angle with TP that HT *produced* does with TQ.

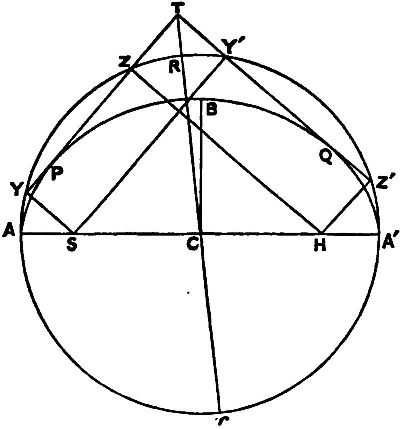
The proposition may also be demonstrated by means of Prop. 7, Cor. 1, chap. III., IV., Prop. 8, chap. 5, and Euc. I. 13, 32.

Cor. It may also be proved that the exterior angle between TP, TQ is half the sum of the angles subtended by PQ at the two foci.

Proposition VIII.

Tangents to an ellipse or hyperbola at right angles to each other intersect on a fixed circle.

Taking the ellipse, let TP, TQ be a pair of tangents at right angles. Draw the auxiliary circle and let it meet TP, TQ and CT at Y, Z, Y', Z', R, r respectively.



Join SY, SY', HZ, HZ'. Then $SYTY'$, $HZ'TZ$ are rectangles (Prop. 10, chap. III., IV.) and $\therefore$ $SY' = TY$ and $HZ' = TZ$.

Also $CT^2 = TR.Tr + CR^2$ (Euc. II. 6)
$= TZ.TY + AC^2$ (Euc. III. 36, Cor.)
$= HZ'.SY' + AC^2$
$= BC^2 + AC^2$ (Prop. 10, chap. III., IV.)
$= AB^2$ (Euc. I. 47).

Consequently T lies on a circle whose centre is C and radius equal to AB.

The proof for the hyperbola is similar, the square on the radius of the circle being $= AC^2 - BC^2$. When AC is less than BC, no tangents can be drawn at right angles to each other, and in the rectangular hyperbola, the asymptotes only are such tangents.

NOTE.—Tangents to a parabola at right angles to each other intersect on the directrix (Prop. 4, chap. I.), and the circle which is the locus of the intersection of tangents to a central conic at right angles to each other is consequently called the *Director Circle.* The director circle of a central conic also possesses another property analogous to the directrix of the parabola; for if P be any point on the conic, and the diameter CP be produced to meet the circle at T, t; and SP, HP be joined, then

$SP.HP = CD^2$ (Prop. 20, chap. III. IV.)
$= \pm AC^2 + BC^2 \mp CP^2$ (Prop. 18, chap. III., IV.)
$= \pm CT^2 \mp CP^2$
$= PT.Pt$ (Euc. II. 5, 6).

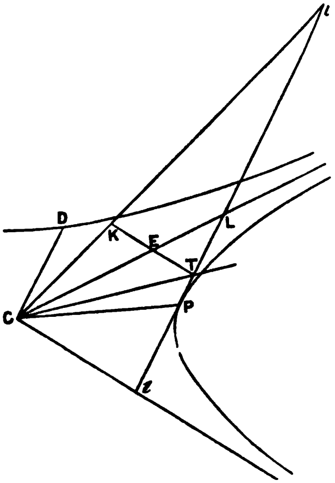
PROPOSITION IX.

If the tangent at any point P on an ellipse or hyperbola meet any pair of conjugate diameters at T, t, then $PT.Pt = CD^2$, CD being the semi-diameter conjugate to CP.

Since the proposition is evidently true in the circle, it follows by projections (Prop. 12, chap. III.) that it is true for the ellipse.

In the case of the hyperbola let the tangent at P meet the asymptotes at L, l. Draw TEK parallel to Cl meeting CL, Ct at E, K respectively.

Then since the line joining the extremities of any pair of conjugate diameters is parallel to one asymptote and bisected by the other (Prop. 15, Cor. chap. IV.), it follows that every line drawn parallel to one asymptote and terminated by a pair of conjugate diameters is bisected by the other asymptote; ∴ TK is bisected at E. Again, since TK is parallel to Cl,



$$\therefore \quad \frac{LT}{Ll}=\frac{TE}{Cl} \quad \text{(Euc. VI. 4).}$$

Similarly $$\frac{tT}{tl}=\frac{TK}{Cl}=\frac{2TE}{Cl};$$

$$\therefore \quad \frac{tT}{tl}=\frac{2LT}{Ll},$$

or $$\frac{Pt-PT}{Pt+PL}=\frac{2(PL-PT)}{2PL}$$ (Since P bisects Ll, Prop. 13, Cor. 1, chap. IV.)

$$=\frac{Pt-PT-(PL-PT)}{Pt+PL-PL}=\frac{Pt-PL}{Pt},$$

or $$\frac{PL-PT}{PL}=\frac{Pt-PL}{Pt},$$

$$\frac{PT}{PL}=\frac{PL}{Pt},$$

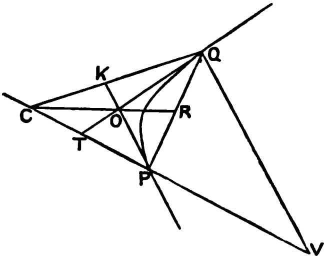
$$\therefore PT.Pt=PL^2=CD^2$$ (Prop. 13, Cor. 1, chap. IV.).

PROPOSITION X.

If the tangent at any point Q on an ellipse or hyperbola meet a diameter CP at T, and QV be the ordinate to this diameter then $CV.CT=CP^2$.

The proposition is manifestly true for the circle, and projecting, it is consequently true for the ellipse; or it may be proved as follows where we consider the case of the hyperbola.

Join PQ, CQ, and let the tangent at P meet CQ, QT at K, O respectively. Join CO and produce it to meet PQ at R; then PQ is bisected at R (Prop. 10, chap. V.).



Hence the triangle $CRP=CRQ$ and $ORP=ORQ$ (Euc. I. 38),

$$\therefore\quad COP=COQ.$$

Again $$\frac{CP}{CT}=\frac{CQP}{CQT}=\frac{COP}{COT}\text{ (Euc. VI. 1)}$$

$$=\frac{CQP-COP}{COQ}=\frac{CQP-COQ}{COP}.$$

Similarly $$\frac{CQ}{CK}=\frac{CQP}{CKP}=\frac{COQ}{COK}=\frac{CQP-COQ}{COP};$$

$$\therefore\quad \frac{CP}{CT}=\frac{CQ}{CK}.$$

Also since QV is parallel to PK (Def. 11, chap. IV.),

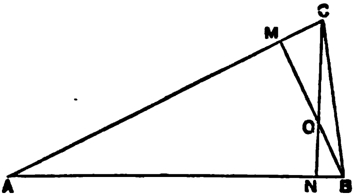
$$\therefore\quad \frac{CV}{CP}=\frac{CQ}{CK}\text{ (Euc. VI. 2)}$$

$$=\frac{CP}{CT},$$

$$\therefore\quad CV.CT=CP^2.$$

PROPOSITION XI.

If a rectangular hyperbola pass through the angular points of a triangle, it also passes through the intersection of perpendiculars from the angles on the opposite sides (the orthocentre).



Let ABC be the triangle. Draw CN perpendicular to AB. and let the hyperbola meet CN at O. Join BO, and produce it to meet AC at M.

Then $AN.NB$ is to $CN.NO$ as the squares on the diameters parallel to AB and CN (Prop. 11, chap. V.).

But these diameters are equal since they are at right angles (Prop. 14, chap. IV.),

$$\therefore\ AN.NB = CN.NO,$$

or
$$\frac{AN}{CN} = \frac{NO}{NB};$$

$\therefore$ the triangles ANC, ONB are equiangular (Euc. VI. 5),

$\therefore$ angle $OBN = ACN$

$= $ complement of CAN,

$\therefore$ AMB is a right angle, and O is the point of intersection of perpendiculars.

MISCELLANEOUS PROBLEMS.

1. Show that in an ellipse $CG.CT=CS^2$.

2. If the tangent at a point P of an ellipse meet the directrices at Q, Q', and S, S' be the corresponding foci, and the other tangents through Q, Q', intersect at T; then the triangles TQS' and $TQ'S$ are similar, and $TQ.TS'=TQ'.TS$.

3. The tangent at a point P of an ellipse meets any two conjugate diameters at T, t, and TS, tH are joined and produced if necessary to meet at Q; prove that the triangles SPT, HPt, TQt are similar.

4. The tangent at a point P of an ellipse meets the auxiliary circle at Y, Z, and the tangents to the circle at Y, Z intersect at T; show that T lies in the ordinate NP produced and that $\frac{NP}{NT}=\frac{BC^2}{AC^2}$. Show also that the locus of T is an ellipse.

5. The tangent at a point P of an ellipse meets the auxiliary circle at Q' to which corresponds Q on the ellipse; prove that the tangent at Q cuts the auxiliary circle at the point corresponding to P.

6. If from a point T on the auxiliary circle, two tangents be drawn to an ellipse touching it at P and Q, and when produced meeting the circle again at p, q; show that the angles PSp and QHq are together equal to the supplement of PTQ, S being the focus nearer P.

7. If a chord be drawn to a series of concentric similar and similarly situated ellipses, and meet one at P and Q, and if on PQ as diameter a circle be described meeting that ellipse again at R, S; show that RS is constant in position.

8. If perpendiculars be drawn from the angular points of a triangle upon the opposite sides, an ellipse can be described touching the sides at the points where the perpendiculars fall. It is required also to construct it.

9. With the intersection of the perpendiculars from the angles of a triangle on the opposite sides as focus, two conics are described touching a side of the triangle and having the other two sides as directrices respectively; prove that their minor axes are equal.

10. From a point T on the auxiliary circle tangents are drawn to an ellipse touching it at P and Q, and meeting the circle again at p, q; show that the angle pCq is equal to the sum of the angles PSQ and PHQ.

11. If PQ, $P'Q'$ be chords of a conic at right angles to one another, and respectively normals to the conic at P, P', show that PP' is parallel to QQ'.

12. PG, the normal to an ellipse at P, is produced to meet CK, a line through C parallel to the tangent at P in the point F; and the ordinate PN produced meets the same line at K; show that

$$\frac{CF.FK}{BC^2} = \frac{FG}{PG}.$$

13. CP, CD are a pair of conjugate semi-diameters of an ellipse or hyperbola, and QV is the ordinate to CP of any point Q on the curve, the tangent at which point meets CP produced at T; if VZ be drawn parallel to PD, meeting CD (produced if necessary) at Z, and PZ, TD be joined, show that PZ and TD are parallel.

14. A straight line is drawn through the angular point A of a triangle ABC to meet the opposite side at a; two points O, O' are taken on it, and CO, CO' meet AB at c, c', and BO, BO' meet CA at b, b': show that a conic passing through a, b, b', c, c' will be touched by CB.

15. Two rectangular hyperbolas intersect at A, B, C, D; show that the circles described on AB and CD as diameters cut orthogonally.

16. A series of ellipses are described having the same foci; at the extremities of either series of latera recta tangents are drawn; show that they all touch a fixed parabola.

17. If SY, SZ be perpendiculars from a focus S on two tangents to an ellipse which intersect at T, the line YZ is perpendicular to the line HT.

18. If the internal and external bisectors of the angle between two tangents drawn to an ellipse meet the axis at G and T, then $CG.CT=CS^2$.

19. Given the lengths of the axes of an ellipse, and the positions of one focus, and of one point in the curve; give a geometrical construction for finding the centre.

20. From the point P on an ellipse lines are drawn to A, A', and from A, A' lines are drawn perpendicular to AP, $A'P$; show that the locus of their intersection will be another ellipse, and find its axes.

21. If from the extremities of any diameter AB of an equilateral hyperbola, lines be drawn to any point P in the curve, they will be equally inclined to the asymptotes.

22. Given a pair of conjugate diameters of a conic, find geometrically the position of the axes (1.) in the case of the hyperbola, (2.) in that of the ellipse.

23. If at the extremities P, Q of any two diameters CP, CQ of an ellipse, two tangents Pp, Qq be drawn cutting each other at T and the diameters produced at p and q, then the areas of the triangles TQp, TPq are equal.

24. Tangents to an ellipse are drawn from any point in a circle through the foci; prove that the lines bisecting the angle between the tangents, all pass through a fixed point.

25. ABC is a triangle inscribed in an ellipse, and the normals at B and C bisect the angles B, C respectively; show that the normal at A bisects the angle A.

26. Perpendiculars are drawn from the foci of a hyperbola on the tangent at a point P intersecting it at Y, Z; perpendiculars are also drawn from the foci of the conjugate hyperbola on the tangent at D intersecting it at Y', Z', (D being the extremity of the diameter conjugate to that passing through P); show that

$$PZ.PY=DY'.DZ'.$$

27. P is a point on an ellipse; the tangent at P meets the axis minor at t, and a perpendicular to CP through P meets the auxiliary circle at Q; show that PQ varies inversely as Ct.

28. $ABCD$ is a quadrilateral, the angles at A and C being equal; a conic is described about $ABCD$ so as to touch the circumscribing circle of ABC at the point B; show that BD is a diameter of the conic.

29. In an ellipse if SY is a perpendicular from the focus S on the tangent at P, and CD conjugate to CP, then

$$SY.CD = SP.BC.$$

30. PQ is a chord of an ellipse normal at P, LCL' the diameter bisecting it, show that PQ bisects the angle LPL', and $LP+PL'$ is constant.

31. If the tangent at any point P of an ellipse cut any pair of conjugate diameters at T, t, and if SP produced cut the circle described round the triangle STt at Q; show that SQ is equal to the axis major of the ellipse.

32. If a tangent be drawn to one of two confocal conics perpendicular to a tangent to the other, prove that the locus of their point of intersection is a conic.

33. If a quadrilateral circumscribe an ellipse, the angles subtended by opposite sides at one of the foci are together equal to two right angles.

34. If the normal at P on an ellipse meet the axis minor at G, and if the tangent at P meet the tangent at the vertex A at V, show that $\frac{SG}{SC} = \frac{PV}{VA}$.

35. ABC is an isosceles triangle of which the side AB is equal to the side AC. BD, BE drawn on opposite sides of BC and equally inclined to it meet AC at D and E. If an ellipse is described round BDE having one of its axes parallel to BC, then AB will be a tangent to the ellipse.

36. If in an ellipse the normal at P meets the axis at G, and a line through P perpendicular to CP meets it at K, then

$$GK.CN = BC^2.$$

37. A tangent to an ellipse meets a pair of conjugate diameters at points equally distant from the centre; show that the locus of such points is a circle.

38. K is the intersection of a normal to an ellipse at a point P with the axis minor; from Q any point in the circle whose centre is K and which passes through the foci S, H, QS, QH are drawn meeting the tangent at P in T, t; show that T and t lie on a pair of conjugate diameters.

39. Two tangents to an ellipse which are at right angles intersect at T, and TR is drawn to touch the auxiliary circle at R; show that TR is equal to the semi-axis minor of the ellipse.

40. An ellipse and hyperbola are described, so that the foci of each are at the extremities of the transverse axis of the other; prove that the tangents at their points of intersection meet the conjugate axis at points equidistant from the centre.

41. Prove that the distance between the two points on the circumference of an ellipse at which a given chord not passing through the centre, subtends the greatest and least angles is equal to the diameter which bisects that chord.

42. The points at which the tangents at the extremities of the transverse axis of an ellipse are cut by the tangent at any point of the curve, are joined, one with each focus; prove that the joining lines intersect on the normal at the point of the curve.

43. A parallelogram is inscribed in an ellipse, and from any point on the ellipse two straight lines are drawn parallel to the sides of the parallelogram; prove that the rectangles under the segments of these straight lines made by the sides of the parallelogram, are to one another in a constant ratio.

44. If the ordinate PN of any point on an ellipse meet the auxiliary circle at Q, and the tangent and normal at P meet the transverse axis at T and G respectively, prove that

$$\frac{TQ}{TP}=\frac{BC}{PG}.$$

45. Two ellipses whose axes are equal, each to each, are placed in the same plane with their centres coincident and axes inclined to each other: draw their common tangents.

46. Find the positions of the foci and directrices of an ellipse which touches at two given points P, Q, two given straight lines PO, QO, and has one focus on the line PQ, the angle POQ being less than a right angle.

47. An ellipse and a hyperbola are so situated that the vertices of each curve are the foci of the other, and the curves are in planes at right angles to each other. If P be a point on the ellipse, and Q a point on the hyperbola, S the vertex, and A the interior focus of that branch of the hyperbola, then

$$PQ+AS=PS+AQ.$$

48. Tangents are drawn from any point on the auxiliary circle to the ellipse; prove that the line joining one of the points of contact with one of the foci is parallel to the line joining the other point of contact with the other focus.

49. Prove that the rectangular hyperbola which has for its foci the foci of an ellipse, will cut the ellipse at the extremities of the equi-conjugate diameters.

50. If EF be one side of a parallelogram described about an ellipse, having its sides parallel to conjugate diameters, and the lines joining E, F to the foci intersect at O, O', show that O, S, O', H lie on a circle.

51. Perpendiculars SY, HY' are drawn from the foci of an ellipse upon a pair of tangents TY, TY'; prove that the angles STY, HTY' are equal or supplementary to the angles at the base of the triangle formed by joining Y, Y' to the centre of the ellipse.

52. If the ordinate at a point P on an ellipse meet the auxiliary circle at Q, prove that CQ produced will always meet the normal at P on a certain circle whose diameter is equal to the sum of the axes of the ellipse.

53. An ellipse of given semi-axes touches three sides of a given rectangle; find its centre and foci.

54. With the ortho-centre of a triangle as centre are described two ellipses, one circumscribing the triangle and the other touching its sides: prove that these ellipses are similar and their homologous axes at right angles.

55. With one focus of a given hyperbola as focus and any tangent to the hyperbola as directrix is described, another hyperbola touching the conjugate axis of the former; prove that the two are similar.

56. If circles be described passing through the centre of a rectangular hyperbola and the points of intersection of a fixed tangent with pairs of conjugate diameters, their centres will lie on a fixed straight line.

57. If the tangent at any point P of a hyperbola cut an asymptote at T, and if SP cut the same asymptote at Q, then $SQ=QT$.

58. Through the middle point of a focal chord of an ellipse a straight line is drawn at right angles to it to meet the axis at R; show that SR bears to SC the duplicate ratio of the chord to the diameter parallel to it.

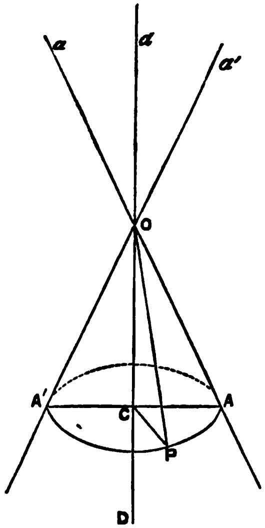
59. Show that if the ordinate of a point P on an ellipse meets the auxiliary circle at Q, and if CQ meets the ellipse at R, then CR is equal to the perpendicular from C on the tangent at P.

CHAPTER VII.

SECTIONS OF A CONE.

Def. The surface traced out by an unlimited straight line, which moves so as always to pass through a fixed point and which makes a constant angle with a fixed straight line passing through that point, is called a *right circular cone.* We shall speak of this surface simply as *a cone.*

The "fixed point" is called the *vertex* of the cone. The "constant angle" is called the *semi-vertical angle.* The "fixed straight line" is called the *axis* of the cone. The moving line in any position is called a *generating line* of the cone.



This definition differs from the one given by Euclid (XI. Def. 18) in two particulars. First, Euclid's cone is the "solid figure," while we define a cone as the bounding "surface." Secondly, Euclid limits the cone by a "base" and the vertex, while the surface in the above definition has unlimited extension on both sides of the vertex.

Let O be the vertex and DOd the axis of a cone, and suppose the plane of the paper to be a section of the cone through the axis, then the boundary of this section is two generating lines AOa, $A'Oa'$ which are equally inclined to OD.

The angle AOA' between these lines is constant, and is called the *vertical angle* of the cone.

PROPOSITION I.

Any plane section of a cone at right angles to the axis is a circle.

Let APA' be such a section whose plane meets the axis at C, and let P be any point on the section. Join CP, OP; then since the angles AOC, POC are equal, and also the angles OCA, OCP, and OC is common to the two triangles ACO, PCO $\therefore$ $CP = CA$ (Euc. I. 26), that is P lies on a circle whose centre is C.

PROPOSITION II.

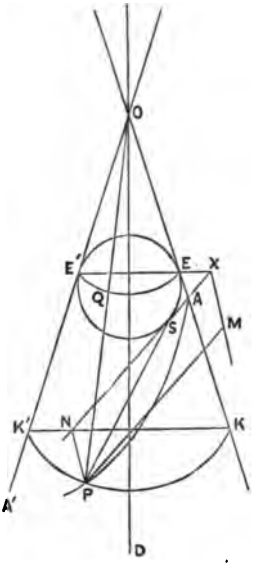
Every plane section of a cone is a conic, according to Definition 1, Introduction.

First,—If the section be at right angles to the axis, we have just shown it to be a circle, which is a particular case of an ellipse whose focus is the centre of the circle, and whose directrix is infinitely distant.

Second,—If the section be through the vertex there are three cases:—(1.) The section may be any pair of generating lines, which is a particular case of the hyperbola whose focus is the point of intersection of the two lines and whose directrix is either of the straight lines bisecting the angle between the two generating lines. (2.) In the limiting position, when the plane of section just touches the cone along a generating line, the section becomes this line which is a particular case of the parabola whose focus is any point in the line, and whose directrix is at right angles to this line through that point. (3.) When the section is simply the vertex: this is the limiting case of a circle or ellipse which has no dimensions.

Third,—Let the section AP cut the axis of the cone obliquely, and be not through the vertex.

Suppose the plane of the paper contain the axis OD of the cone and be at right angles to the plane of section AP, and let AN be the line of intersection of the plane section with the plane of the paper.



Inscribe a sphere $EQE'S$ in the cone so as to touch the plane section at S, which will obviously be in the line AN. The cone will touch this sphere along a circle EQE'. Let XM be the intersection of the plane of this circle with the plane section AP; then XM is at right angles to the plane of the paper, and so at right angles to XAN. Draw PN perpendicular to AN and so parallel to XM, and let KPK' be a section of the cone through PN and at right angles to the axis. Draw PM perpendicular to XM, and join PS, PO, the latter meeting the circular section EQE' at Q.

Then $OK=OP$, and $OE=OQ$;

$\therefore$ $EK=PQ=PS$ (since PQ, PS are tangents to the sphere).

Again, since NK is parallel to EX,

$$\therefore \frac{EK}{NX}=\frac{EA}{AX} \quad \text{(Euc. VI. 2),}$$

$$\text{or } \frac{SP}{PM}=\frac{AS}{AX} \quad \text{(Euc. I. 34, and since } AE, AS \text{ are tangents);}$$

and hence P lies on a conic whose focus is S and directrix XM.

The conic will be a parabola, ellipse, or hyperbola, according
as AS or AE is $=, <$ or $> AX$,
as angle AXE is $=, <$ or $> AEX$ (Euc. I. 5 and 18),
as $AXE+AEX$ is $=, <$ or $> 2AEX$,
as EAS is $=, <$ or $> OEE'+OE'E$ (Euc. I. 32, 15, 5),
as OAN is $=, <$ or $>$ 2 right angles$-EOE'$ (Euc. I. 32),
as $OAN+AOA'$ is $=, <$ or $>$ 2 right angles.

Hence the conic will be a parabola if AN is parallel to OA', that is if the section be parallel to a generating line; as might easily have been seen, for the curve then consists of one branch which extends indefinitely from A.

The conic will be an ellipse if AN meets OA' on the same side of O as A, that is if the section meet all the generating lines on the same side of the vertex of the cone. This might have been seen independently, for we have then a closed curve, which being a conic must be an ellipse.

The conic will be a hyperbola if AN meets $A'O$ produced through the vertex, that is if the section meet the cone at both sides of the vertex. This also might have been seen independently, as the conic then consists of two infinite branches, and must therefore be a hyperbola.

It will have been observed that in the figure we only represent that portion of the cone which is on one side of the plane of the paper.

In the case of the ellipse or hyperbola two spheres may be inscribed in the cone so as to touch the plane section: the point of contact for the second sphere in each case is the other focus of the section.

PROBLEMS.

1. When a section of a cone is made by a plane which is not parallel to a generating line, and two spheres be inscribed in the cone so as to touch the plane, show directly that the sum or difference (as the case may be) of any point on the section from the two points where the plane is touched by the spheres is constant, being equal to the portion of any generating line of the cone intercepted between the circles of contact of the two spheres.

2. Prove that if a right circular cylinder be cut by a plane the section is an ellipse.

3. In any hyperbolic section of a right cone, the angle between the asymptotes never exceeds the angle of the cone.

4. If AA' be the axis major of an elliptic section of a right cone, AQ, $A'Q'$ the diameters of the circular sections of the cone drawn through A, A', and BB' the axis minor of the ellipse, then $BB'^2 = AQ.A'Q'$.

5. If AA' be the major axis of an elliptic section of a cone, and O the vertex, show that the square on the minor axis of the section is proportional to the area of the triangle OAA'.

6. If V be the vertex of a right cone and AA' the major axis of an elliptic section, the difference between VA and VA' is equal to the distance between the foci.

7. Given a right cone and a point within it, there are but two sections which have this point for focus; and the planes of these sections make equal angles with the straight line joining the given point and the vertex of the cone.

8. Show how to cut a given cone so that the section may be a parabola of given latus rectum.

9. Give a geometrical construction by which a cone may be cut, so that the section may be an ellipse of given eccentricity.

10. Find how an obtuse-angled right cone must be cut by a plane that the section may be an equilateral hyperbola.

11. Show how to cut a right cone so that the section may be an ellipse whose axes are of given length.

12. Find the position of the vertex and axis of a cone of given vertical angle in order that a given parabola may be a section of the cone.

13. The foci of all parabolic sections which can be cut from a given right cone lie upon the surface of another right cone.

14. If two plane sections of a right cone be taken, having the same directrix, the foci corresponding to that directrix lie on a straight line which passes through the vertex.

15. Prove that the locus of the vertices of all right cones of which a given fixed conic is a section is another conic, such that the vertices of all the right cones of which it is a section lie on the given conic.

16. Different elliptic sections of a right cone are taken such that their minor axes are equal; show that the locus of their centres is a hyperboloid of revolution, having the cone asymptotic to it.

17. If two cones be described touching the same two spheres, the eccentricities of the two sections of them made by the same plane bear to one another a ratio constant for all positions of the plane.

18. Two cones whose vertical angles are supplementary, are placed with their vertices coincident and their axes at right angles, and are cut by a plane perpendicular to the common generating line; prove that the directrices of the sections of one cone pass through the foci of the section of the other.

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